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# Computer Formulations of Aircraft Models for Simulation Studies



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## SYMBOLS

| | |
|---|---|
| $A$ | angle of attack |
| $A_j$ | linear acceleration component |
| $B$ | angle of sideslip |
| $C_i$ | direction cosine rate parameter |
| $C_K$ | $K$th control surface |
| $CF_i$ | $i$th control force |
| $CM_i$ | $i$th control moment |
| $D_{F_n,C_K}$ | derivative of the $n$th aerodynamic force component with respect to the $K$th control surface |
| $D_{i,j}$ | direction cosine |
| $D_{F_i,U_j}$ | derivative of the $i$th aerodynamic force component with respect to the $j$th component of linear velocity |
| $D_{F_i,P_j}$ | derivative of the $i$th aerodynamic force component with respect to the $j$th component of angular velocity |
| $D_{F_i,A_j}$ | derivative of the $i$th aerodynamic force component with respect to the $j$th component of linear acceleration |
| $D_{M_i,U_j}$ | derivative of the $i$th aerodynamic moment component with respect to the $j$th component of linear velocity |
| $D_{M_i,P_j}$ | derivative of the $i$th aerodynamic moment component with respect to the $j$th component of angular velocity |
| $F^i(X)$ | force component in wind-tunnel axes |
| $F^i(Y)$ | force component in body axes |
| $FG_i$ | $i$th component of gravitational force |
| $FDU_i$ | $i$th component of aerodynamic force due to linear velocity perturbations |
| $FDP_i$ | $i$th component of aerodynamic force due to angular velocity perturbations |
| $FR_i$ | $i$th component of the inertia force |

| | |
|---|---|
| $FT_i$ | $i$th component of thrust |
| $\bar{g}$ | gravity vector |
| $\bar{H}$ | angular momentum vector |
| $IM_j$ | $j$th component of the inertia moment |
| $J_{ij}$ | a component of the inertia tensor |
| $K_n$ | angle of elevation of the $n$th thrust vector |
| $L_{in}$ | $i$th component of the position vector of the point of application of the $n$th thrust vector |
| $MDU_i$ | $i$th component of the aerodynamic moment due to linear velocity perturbations |
| $MDP_i$ | $i$th component of the aerodynamic moment due to angular velocity perturbations |
| $P_i$ | $i$th component of the angular velocity vector |
| $[R_1]$, $[R_2]$,$[R_3]$ | rotation matrices |
| $SF_i$ | $i$th component of the static aerodynamic force in body axes |
| $S_{F_n}$ | $n$th component of static aerodynamic force in wind-tunnel axes |
| $SM_i$ | $i$th component of static aerodynamic moment in body axes |
| $S_{M_n}$ | $n$th component of static aerodynamic moment in wind-tunnel axes |
| $TD_{F_i,U_j}$ | transformed aerodynamic stability derivative of the $i$th aerodynamic force component with respect to the $j$th component of linear velocity |
| $TD_{I,C}$ | $i$th component of the transformed control derivative |
| $TD_{F_i,P_j}$ | transformed aerodynamic stability derivative of the $i$th aerodynamic force component with respect to the $j$th component of angular velocity |
| $TD_{F_i,A_j}$ | transformed aerodynamic stability derivative of the $i$th aerodynamic force component with respect to the $j$th component of linear acceleration |
| $TD_{M_i,U_j}$ | transformed aerodynamic stability derivative of the $i$th aerodynamic moment component with respect to the $j$th component of linear velocity |
| $TD_{M_i,P_j}$ | transformed aerodynamic stability derivative of the $i$th aerodynamic moment component with respect to the $j$th component of angular velocity |

| | |
|---|---|
| $TM_i$ | $i$th component of the thrust moment |
| $U_i$ | $i$th component of the linear velocity vector |
| $\bar{V}$ | velocity vector |
| $X^i$ | wind-tunnel coordinates |
| $Y^i$ | body coordinates |
| $\Delta A_i$ | linear acceleration increments |
| $\Delta C_K$ | control increments |
| $\Delta P_i$ | angular velocity increments |
| $\Delta U_i$ | linear velocity increments |
| $\bar{\omega}$ | angular velocity vector |

Subscripts

| | |
|---|---|
| $E$ | earth-fixed coordinate system |
| $F$ | forces |
| $i$ | components of aerodynamic forces and moments, gravity forces, inertia forces, thrust forces and moments |
| $M$ | moments |
| $n$ | force and moment components |
| $o$ | initial values |

Superscripts

| | |
|---|---|
| $\alpha$ | identifying indices and indices of contravariance |
| $\beta$ | identifying indices and indices of contravariance |
| $i$ | identifying indices and indices of contravariance |
| $j$ | identifying indices and indices of contravariance |

# COMPUTER FORMULATIONS OF AIRCRAFT MODELS FOR SIMULATION STUDIES

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## SUMMARY

A mathematical model of a dynamical system describes the physical characteristics of the system and can be used to determine the response of the system to the forces encountered. In the case of aeronautical systems exposed to gravity, inertia, aerodynamic, and thrust forces, the mathematical model enables the engineer to determine the state of the vehicle, its spatial orientation, and the geographical location as a function of time. Sometimes the vehicle response is determined by direct calculation; at other times, simulators are used. The simulator moves in response to the computed solutions of the mathematical model equations. These solutions represent the response of the system to the forcing functions generated by the simulator pilot's control inputs. The formulation of mathematical models of complex multidegrees-of-freedom dynamical systems is time consuming and subject to human error. In view of the complexity involved, it is desirable to mechanize as much of the formulation as possible. Recent developments in formula manipulation compilers have led to an extension of the area of application of digital computers beyond the purely numerical data processing stage. These developments, combined with the design of several symbol manipulation languages, enable computers to be used for symbolic mathematical computation. This technique provides for the symbolic manipulation of mathematical expressions: for example, the expression SIN(X) can be differentiated, resulting in the expression COS(X). Moreover, it can be used frequently to obtain symbolic solutions in problem areas that heretofore could only be approached numerically. A computer system and language that can be used to perform symbolic manipulations in an interactive mode have been used to formulate a mathematical model of an aeronautical system. The example demonstrates that once the procedure is established, the formulation and modification of models for simulation studies can be reduced to a series of routine computer operations.

## INTRODUCTION

Excluding the control loops, the mathematical description of an aeronautical system requires at least 12 equations (ref. 1): 3 force equations; 3 moment equations; 3 Euler angle equations or 9 direction cosine equations to determine the spatial orientation of the vehicle, and 3 equations to determine the geographical location of the vehicle in inertial space. In view of this complexity, it is desirable to mechanize as much of the formulation as possible.

Research undertaken with the object of mechanizing the formulation of mathematical models of aeronautical systems, has directed attention to the use of digital computers for symbolic mathematical computation. To date, the majority of computer systems and languages has been developed to facilitate the processing of numerical data in one form or another. More recently, however, a variety of symbol or formula-manipulation languages has evolved. The choice of system and language to be used for a given purpose depends on accessibility, personal preference, the type and magnitude of the problems to be formulated, and the computer facilities available to the user. At the time of writing, the two most important contenders in the symbol manipulation field appeared to the author to be REDUCE and MACSYMA. REDUCE is a program designed for general algebraic computations of interest to mathematicians, physicists, and engineers. In addition to the usual algebraic manipulations, it has the capability of performing calculations of special interest to high energy physicists. Originally, it began as a system for solving special problems that arise in high energy physics, where much tedious repetitive calculation is involved. However, it was quickly recognized that the computer processes being used were quite general, and could be used for a great variety of algebraic manipulations. Although REDUCE can operate in a batch processing mode, it is intended primarily for interactive calculations in a time shared environment. Hence, it is command oriented, rather than program oriented, since the result of a given command may be required before proceeding to the next step. REDUCE is available on most IBM 360 or 370 series computers, the DEC PDP-10, and the CDC 6400, 6500, 6600, and 7600 machines. At the time of writing, the MACSYMA system was available only at MIT through the Advanced Research and Project Agency (ARPA) Network (ref. 2). It is a large computer programming system, which can be used to perform symbolic as well as numerical mathematical operations. It was developed by the Mathlab group of project MAC's Automatic Programming Division specifically for interactive use. In addition to manipulating algebraic expressions, the MACSYMA system can differentiate, integrate, take limits, solve systems of linear or polynomial equations and factor polynomials, expand functions, plot curves, and manipulate matrices. Moreover, it is continuously evolving to meet the needs of users. In view of its flexibility and diverse capabilities, MACSYMA was the system chosen to formulate mathematical models of aeronautical systems for simulation purposes. Subsequent experience with the system confirmed the wisdom of this choice. Of special interest to users is the facility with which MACSYMA can derive special forms from a more general formulation. The same feature permits users to introduce new sets of system parameters or modify existing ones as the occasion demands. It will be seen that a simple programming statement can be used to introduce a new set of inertia tensors, static aerodynamic coefficients, control force derivatives, aerodynamic stability derivatives, or thrust coefficients to meet model or design changes. An important aspect of the formulation of mathematical models of aeronautical systems for simulation and other purposes is the specification of the system of forces and moments. In aeronautical applications, the thrust and inertia forces and moments, and the gravity force can be formulated without difficulty; but the aerodynamic forces and moments require more detailed consideration. These are represented by the static forces and moments, the control forces, and the perturbation forces that depend on the aerodynamic stability derivatives. These forces and moments have to be transformed from wind or wind-tunnel stability axes to aircraft body axes before the formulation can proceed. Although these formulations and transformations are not complicated, they are complex and unwieldy and are likely to contain errors when performed manually. The interactive capability, versatility, and simplicity of the MACSYMA system make it attractive to programmers and nonprogrammers alike. To illustrate these aspects of the system, a mathematical model of an aeronautical system has been formulated and subjected to a series of modifications.

# ANALYSIS

## Transformation Laws

A necessary preliminary to the formulation of mathematical models of aeronautical systems is the transformation of static aerodynamic forces and moments, control force derivatives, and the aerodynamic stability derivatives from wind or wind-tunnel stability axes to aircraft body axes. It will be seen that whereas the static forces and moments obey the same transformation law as the system coordinates, the aerodynamic stability derivatives transform like the components of a mixed tensor, having one index of covariance and one index of contravariance (ref. 3). Moreover, because of the equivalence of covariant and contravariant transformations in orthogonal Cartesian systems of coordinates, the transformations can be treated as doubly covariant or doubly contravariant, if this simplifies the formulation. The rule for transforming static force coefficients from the $X$ frame of reference (the wind-tunnel axes system) to the $Y$ frame (the body axes system) is obtained as follows.

Since all vectors, including the position vector of a point, obey the same transformation law, it follows that the force and moment vectors obey the same transformation law as the system coordinates; that is, if $S_{F_n}$ denotes a static aerodynamic force in the $X$ frame of reference, and $SF_i$ denotes the corresponding transformed force in the $Y$ reference frame, then since the transformation law for coordinates is

$$Y^i = \frac{\partial Y^i}{\partial X^n} X^n \tag{1}$$

it follows that

$$SF_i = \frac{\partial Y^i}{\partial X^n} S_{F_n} \tag{2}$$

where $Y = Y(X)$ remains to be specified.

Likewise, if $D_{F_n,C_K}$ denotes the $n$th control force derivative with respect to the $K$th control surface, as measured in the $X$ reference frame, and $TD_{i,c}$ denotes the corresponding transformed derivative in the $Y$ frame, then

$$TD_{i,c} = \frac{\partial Y^i}{\partial X^n} D_{F_n,C_K} \tag{3}$$

The aerodynamic stability derivatives measure the rates of change of aerodynamic forces and moments with respect to motion vector components. In keeping with the usual practice in aerodynamic formulations, motion vector components will refer specifically to components of the linear velocity vector, components of the angular velocity vector, and components of the corresponding linear and angular acceleration vectors. The transformation law for these derivatives may be obtained as follows:

Let $F^i(Y)$ be a force or moment in the $Y$ system of axes, and let $U^j(Y)$ be a motion vector component in this system of axes.

Similarly, let $F^\alpha(X)$ be a force or moment in the $X$ system of axes, and let $U^\beta(X)$ be a motion vector component in this system of axes.

Then the stability derivatives with respect to motion components, as measured in the $Y$ system of axes, are related to the corresponding derivatives in the $X$ system of axes by the following equation:

$$\frac{\partial F^i(Y)}{\partial U^j(Y)} = \frac{\partial F^i(Y)}{\partial F^\alpha(X)} \frac{\partial F^\alpha(X)}{\partial U^\beta(X)} \frac{\partial U^\beta(X)}{\partial U^j(Y)} \tag{4}$$

Force, moment, and motion vector components obey the same transformation law as the system coordinates, that is, since

$$Y^i = \frac{\partial Y^i}{\partial X^\alpha} X^\alpha$$

$$F^i(Y) = \frac{\partial Y^i}{\partial X^\alpha} F^\alpha(X) \tag{5}$$

and

$$U^j(Y) = \frac{\partial Y^j}{\partial X^\beta} U^\beta(X) \tag{6}$$

it follows that

$$U^\beta(X) = \frac{\partial X^\beta}{\partial Y^j} U^j(Y) \tag{7}$$

Substitution from equations (5) and (7) in equation (4) gives

$$\frac{\partial F^i(Y)}{\partial U^j(Y)} = \left(\frac{\partial Y^i}{\partial X^\alpha} \frac{\partial X^\beta}{\partial Y^j}\right) \frac{\partial F^\alpha(X)}{\partial U^\beta(X)} \tag{8}$$

In these and subsequent equations the summation convention is assumed; that is, if in any term an index occurs twice, the term is to be summed with respect to that index for all admissible values of the index.

Equation (8) shows that the aerodynamic stability derivatives transform like the components of a mixed tensor, having one index of covariance and one index of contravariance. Being a tensor of rank 2, equation (8) represents 9 equations, with each equation having, in general, 9 terms.

A disadvantage of the transformation law as formulated in equation (8) is the requirement that the transformation equation $X = X(Y)$ must be used. Fortunately, it is possible to avoid the use of the inverse transformation $X = X(Y)$ if the coordinate transformations are orthogonal Cartesian. It can be shown that for orthogonal Cartesian transformations

$$\frac{\partial Y^i}{\partial X^j} = \frac{\partial X^j}{\partial Y^i} \tag{9}$$

Substitution of this relationship in equation (8) yields

$$\frac{\partial F^i(Y)}{\partial U^j(Y)} = \left(\frac{\partial Y^i}{\partial X^\alpha}\,\frac{\partial Y^j}{\partial X^\beta}\right)\frac{\partial F^\alpha(X)}{\partial U^\beta(X)} \tag{10}$$

The form of equation (10) shows that if the transformations are orthogonal Cartesian, the aerodynamic stability derivatives can be treated as doubly contravariant tensors. This form has the advantage that the inverse transformation $X = X(Y)$ is no longer required.

Although the notation used in equations (8) and (10) reveals the tensor character of the transformation, superscripts will not be used in subsequent work. They will be replaced by subscripts, which are more convenient for programming purposes. For the same reason, the Greek symbols $\alpha$ and $\beta$ will be replaced by the letters $M$ and $N$. The following definitions are required:

$$\frac{\partial F^\alpha(X)}{\partial U^\beta(X)} = \frac{\partial F_\alpha(X)}{\partial U_\beta(X)} = \frac{\partial F_M(X)}{\partial U_N(X)} = D_{F_M, U_N} \tag{11}$$

and

$$\frac{\partial F^i(Y)}{\partial U^j(Y)} = \frac{\partial F_i(Y)}{\partial U_j(Y)} = TD_{F_i, U_j} \tag{12}$$

where $D_{F_M, U_N}$ denotes the derivative of the $M$th component of aerodynamic force, with respect to the $N$th component of the motion vector; and $TD_{F_i, U_j}$ denotes the transformed derivative of the $i$th component of aerodynamic force, with respect to the $j$th component of the motion vector.

## Aeronautical Reference Systems

There are many coordinate systems in use in aeronautical research. Aerodynamic data obtained from wind-tunnel experiments may be referred to wind axes or to wind-tunnel stability axes. When the wind axes are used, the $X_1$ axis is aligned with the relative wind at all times. Most wind-tunnel data are referred to the wind-tunnel stability axes system. For this system, the $X_1$ axis is in the same horizontal plane as the relative wind at all times. In addition to the wind axes and the wind-tunnel stability axes, there are other systems of axes fixed in the body and moving with the body. These are referred to as body axes. In aerospace applications, a body axis system has the $Y_1$ axis fixed along the longitudinal centerline of the body, the $Y_2$ axis normal to the plane of symmetry, and the $Y_3$ axis in the plane of symmetry. The equations of motion of aerospace vehicles are formulated with respect to body axes. The main advantage of these axes in motion calculations is that vehicle moments and products of inertia about the axes are constants. When the body axes are chosen so that the products of inertia vanish, they are known as principal axes. A system of axes, which is frequently used to study the stability of aircraft in the presence of disturbing forces that produce small perturbations, is the flight stability axes. This is an orthogonal system fixed to the vehicle, the $Y_1$ axis of which is aligned with the relative wind vector when the vehicle is in a steady-state condition, but then rotates with the vehicle after a disturbance as the vehicle changes angle of attack and sideslip. Some of these axes are shown in figure 1 (ref. 4).

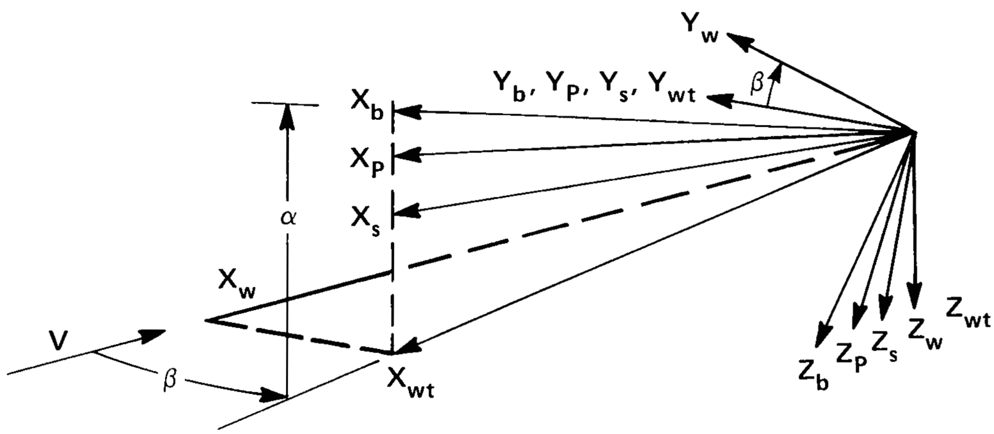



Figure 1.– Systems of reference axes, including body, principal, wind, flight stability, and wind-tunnel stability.

## Transformation Equations

The elements of the matrices defining a transformation from wind or wind-tunnel stability axes to body axes are functions of the angle of attack ($A$) and the angle of sideslip ($B$). Moreover, coordinates in wind-tunnel axes are denoted by a column vector of coordinates $X_i$, and the body axes coordinates by a column vector $Y_i$. To bring a reference frame from the wind axes into coincidence with the body axes involves a negative rotation ($B$) about the $Y_3$ axis, followed by a positive rotation ($A$) about the $Y_2$ axis. These matrices may be entered and multiplied when communication has been established and the system prints (C1). When this occurs, the user types ENTERMATRIX(m,n) which allows one to enter a matrix, element by element, with MACSYMA requesting values for each of the (m,n) entries as follows:

```
(C1) ENTERMATRIX(3,3);

ROW 1 COLUMN 1  COS(A);

ROW 1 COLUMN 2  0;

ROW 1 COLUMN 3 -SIN(A);

ROW 2 COLUMN 1  0;

ROW 2 COLUMN 2  1;

ROW 2 COLUMN 3  0;

ROW 3 COLUMN 1  SIN(A);

ROW 3 COLUMN 2  0;

ROW 3 COLUMN 3  COS(A);

MATRIX-ENTERED

                                   [ COS(A)  0  - SIN(A) ]
                                   [                     ]
(D1)                               [   0     1     0     ]
                                   [                     ]
                                   [ SIN(A)  0   COS(A)  ]

(C2) ENTERMATRIX(3,3);


ROW 1 COLUMN 1  COS(B);

ROW 1 COLUMN 2  -SIN(B);

ROW 1 COLUMN 3  0;

ROW 2 COLUMN 1  SIN(B);

ROW 2 COLUMN 2  COS(B);

ROW 2 COLUMN 3  0;

ROW 3 COLUMN 1  0;

ROW 3 COLUMN 2  0;

ROW 3 COLUMN 3  1;
```

MATRIX-ENTERED

(D2) $$\begin{bmatrix} COS(B) & - SIN(B) & 0 \\ SIN(B) & COS(B) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(C3) ENTERMATRIX(3,1);

ROW 1 COLUMN 1  X[1];

ROW 2 COLUMN 1  X[2];

ROW 3 COLUMN 1  X[3];

MATRIX-ENTERED

(D3) $$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

(C4) (D1).(D2).(D3);

(D4) $$\begin{bmatrix} COS(A)\ (X_1\ COS(B) - X_2\ SIN(B)) - X_3\ SIN(A) \\ X_1\ SIN(B) + X_2\ COS(B) \\ SIN(A)\ (X_1\ COS(B) - X_2\ SIN(B)) + X_3\ COS(A) \end{bmatrix}$$

(C5) FOR I:1 THRU 3 DO ROW[1]:FIRST(ROW((D4),I))$

(C6) FOR I:1 THRU 3 DO (Y[I]:ROW[I][1],DISPLAY(Y[I]));

(D6)

$$Y_1 = COS(A)\ (X_1\ COS(B) - X_2\ SIN(B)) - X_3\ SIN(A)$$

$$Y_2 = X_1\ SIN(B) + X_2\ COS(B)$$

$$Y_3 = SIN(A)\ (X_1\ COS(B) - X_2\ SIN(B)) + X_3\ COS(A)$$

In order to more fully appreciate the results obtained so far, the reader should note that MACSYMA requests the $i$th row and the $j$th column of the matrix being entered by typing ROWICOLUMNJ. The user merely provides the corresponding element. When all $m \times n$ elements have been entered, the system types MATRIX-ENTERED, formulates the matrix and assigns an identifying number (DI). When the user types the command (C4), that is, (D1).(D2).(D3), the three matrices are multiplied in the order requested and the product matrix is displayed in (D4).

The two programming steps shown in (C5) and (C6) lead to the functional form (D6), which represents the required transformation from wind axes $X_I$ to body axes $Y_J$.

## FORCES

### Transformation of Static Forces

The static aerodynamic forces transform like the components of a contravariant vector; that is, if $S_{F_n}$ denotes a static aerodynamic force in the $X$ frame of reference, and $SF_i$ denotes the corresponding transformed force in the $Y$ reference frame, then from equation (2)

$$SF_i = \frac{\partial Y^i}{\partial X^n} S_{F_n} \tag{2}$$

where $Y = Y(X)$ is obtained from the displayed output (D6).

Given the transformation equations (D6), the transformed aerodynamic static forces are obtained by expanding equation (2). Three programming steps are sufficient to formulate the required values. The simple program and the displayed results are

```
(C7) SF[I]:=0$

(C8) FOR I:1 THRU 3 DO FOR M:1 THRU 3 DO
 SF[I]:SF[I]+DIFF(Y[I],X[M])*S[F[M]]$

(C9) FOR I:1 THRU 3 DO DISPLAY(SF[I]);
```

$$SF_1 = - S_{F_2} \cos(A) \sin(B) + S_{F_1} \cos(A) \cos(B) - S_{F_3} \sin(A)$$

$$SF_2 = S_{F_1} \sin(B) + S_{F_2} \cos(B)$$

$$SF_3 = - S_{F_2} \sin(A) \sin(B) + S_{F_1} \sin(A) \cos(B) + S_{F_3} \cos(A)$$

(D9) DONE

Transformation of Control Force Derivatives

The control force derivatives obey the same transformation law as the static forces; that is, if $D_{F_n,C_K}$ denotes the $n$th control force derivative with respect to the $K$th control surface as measured in the $X$ reference frame, and $TD_{I,C}$ denotes the corresponding transformed derivative in the $Y$ frame, then using equation (3)

$$TD_{i,c} = \frac{\partial Y^i}{\partial X^n} D_{F_n,C_k} \tag{3}$$

where $Y = Y(X)$ is again obtained from the displayed output (D6).

As in the preceding section, the transformed control derivatives are obtained by expanding the transformation law for derivatives given the transformation equations (D6). The transformed derivatives are obtained by executing the following simple program, which has exactly the same form as the program used to transform the static forces. These are:

```
(C10) TD[I,C]:=0$

(C11) FOR I:1 THRU 3 DO FOR M:1 THRU 3 DO
 TD[I,C]:TD[I,C]+DIFF(Y[I],X[M])*D[F[M],C[K]]$

(C12) FOR I:1 THRU 3 DO DISPLAY(TD[I,C]);
```

$$TD_{1,C} = -D_{F_2,C_K} COS(A)SIN(B) + D_{F_1,C_K} COS(A)COS(B) - D_{F_3,C_K} SIN(A)$$

$$TD_{2,C} = D_{F_1,C_K} SIN(B) + D_{F_2,C_K} COS(B)$$

$$TD_{3,C} = -D_{F_2,C_K} SIN(A)SIN(B) + D_{F_1,C_K} SIN(A)COS(B) + D_{F_3,C_K} COS(A)$$

(D12) DONE

The corresponding control forces are obtained by multiplying the control derivatives by the appropriate control increments $\Delta C_K$. The following two programming steps are sufficient to ensure evaluation of the required forces. These are denoted by $CF_i$ in the displayed output.

```
(C13) FOR I:1 THRU 3 DO CF[I]:TD[I,C]*DEL(C[K])$

(C14) FOR I:1 THRU 3 DO DISPLAY(CF[I]);
```

$$CF_1 = (-D_{F_2,C_K} COS(A)SIN(B) + D_{F_1,C_K} COS(A)COS(B) - D_{F_3,C_K} SIN(A))\ DEL(C_K)$$

$$CF_2 = (D_{F_1,C_K} \mathrm{SIN}(B) + D_{F_2,C_K} \mathrm{COS}(B))\mathrm{DEL}(C_K)$$

$$CF_3 = (-D_{F_2,C_K} \mathrm{SIN}(A)\mathrm{SIN}(B) + D_{F_1,C_K} \mathrm{SIN}(A)\mathrm{COS}(B) + D_{F_3,C_K} \mathrm{COS}(A))\ \mathrm{DEL}(C_K)$$

```
(D14)                          DONE
```

Forces Produced by Linear Velocity Perturbations

The next step in the formulation involves the determination of the aerodynamic forces produced when an aircraft is subjected to linear velocity perturbations $\Delta U_j$. Before these forces can be determined, the aerodynamic stability derivatives, with respect to linear velocity components, must be transformed from wind or wind-tunnel stability axes to aircraft axes. For a detailed discussion of the transformation of these derivatives, the reader is referred to equations (4) through (10). In this application, the aerodynamic stability derivatives of the $i$th force with respect to the $j$th velocity components are denoted by $D_{F_i,U_j}$. The corresponding transformed derivatives are denoted by $TD_{F_i,U_j}$. The program required for this application assumes the form

```
(C15) TDU[I,J]:=0$

(C16) FOR I:1 THRU 3 DO FOR J:1 THRU 3 DO
FOR M:1 THRU 3 DO FOR N:1 THRU 3 DO
 TDU[I,J]:TDU[I,J]+DIFF(Y[I],X[M])*DIFF(Y[J],X[N])*D[F[M],U[N]]$
```

It only remains to multiply the transformed derivatives by the appropriate velocity increments to obtain the required forces, which are denoted by $FDU_i$. The next three programming steps instruct MACSYMA to evaluate and display the forces produced by linear velocity perturbations. These are

```
(C17) FDU[I]:=0$

(C18) FOR I:1 THRU 3 DO FOR J:1 THRU 3 DO
 FDU[I]:FDU[I]+TDU[I,J]*DEL(U[J])$

(C19) FOR I:1 THRU 3 DO DISPLAY(FDU[I]);
```

$$FDU_1 = (D_{F_2,U_2} \mathrm{COS}(A)\mathrm{SIN}(A)\mathrm{SIN}^2(B)$$

$$- D_{F_2,U_1} \mathrm{COS}(A)\mathrm{SIN}(A)\mathrm{COS}(B)\mathrm{SIN}(B)$$

$$- D_{F_1,U_2} \mathrm{COS}(A)\mathrm{SIN}(A)\mathrm{COS}(B)\mathrm{SIN}(B) + D_{F_3,U_2} \mathrm{SIN}^2(A)\mathrm{SIN}(B)$$

$$- D_{F_2,U_3} \cos^2(A)\sin(B) + D_{F_1,U_1} \cos(A)\sin(A)\cos^2(B)$$

$$-D_{F_3,U_1} \sin^2(A)\cos(B) + D_{F_1,U_3} \cos^2(A)\cos(B)$$

$$- D_{F_3,U_3} \cos(A)\sin(A))DEL(U_3)$$

$$+ (-D_{F_2,U_1} \cos(A)\sin^2(B) - D_{F_2,U_2} \cos(A)\cos(B)\sin(B)$$

$$+ D_{F_1,U_1} \cos(A)\cos(B)\sin(B) - D_{F_3,U_1} \sin(A)\sin(B)$$

$$+ D_{F_1,U_2} \cos(A)\cos^2(B) - D_{F_3,U_2} \sin(A)\cos(B))DEL(U_2)$$

$$+ (D_{F_2,U_2} \cos^2(A)\sin^2(B) - D_{F_2,U_1} \cos^2(A)\cos(B)\sin(B)$$

$$- D_{F_1,U_2} \cos^2(A)\cos(B)\sin(B) + D_{F_3,U_2} \cos(A)\sin(A)\sin(B)$$

$$+ D_{F_2,U_3} \cos(A)\sin(A)\sin(B) + D_{F_1,U_1} \cos^2(A)\cos^2(B)$$

$$- D_{F_3,U_1} \cos(A)\sin(A)\cos(B) - D_{F_1,U_3} \cos(A)\sin(A)\cos(B)$$

$$+ D_{F_3,U_3} \sin^2(A))DEL(U_1)$$

$$FDU_2 = (-D_{F_1,U_2} \sin(A)\sin^2(B) - D_{F_2,U_2} \sin(A)\cos(B)\sin(B)$$

$$+ D_{F_1,U_1} \sin(A)\cos(B)\sin(B) + D_{F_1,U_3} \cos(A)\sin(B)$$

$$+ D_{F_2,U_1} \sin(A)\cos^2(B) + D_{F_2,U_3} \cos(A)\cos(B))DEL(U_3)$$

$$+ (D_{F_1,U_1} \sin^2(B) + D_{F_2,U_1} \cos(B)\sin(B) + D_{F_1,U_2} \cos(B)\sin(B)$$

$$+ D_{F_2,U_2} \cos^2(B))DEL(U_2) + (-D_{F_1,U_2} \cos(A)\sin^2(B)$$

$$- D_{F_2,U_2} \cos(A)\cos(B)\sin(B) + D_{F_1,U_1} \cos(A)\cos(B)\sin(B)$$

$$- D_{F_1,U_3} \text{SIN(A)SIN(B)} + D_{F_2,U_1} \text{COS(A)COS}^2\text{(B)}$$

$$- D_{F_2,U_3} \text{SIN(A)COS(B))DEL}(U_1)$$

$$FDU_3 = (D_{F_2,U_2} \text{SIN}^2\text{(A)SIN}^2\text{(B)} - D_{F_2,U_1} \text{SIN}^2\text{(A)COS(B)SIN(B)}$$

$$- D_{F_1,U_2} \text{SIN}^2\text{(A)COS(B)SIN(B)} - D_{F_3,U_2} \text{COS(A)SIN(A)SIN(B)}$$

$$- D_{F_2,U_3} \text{COS(A)SIN(A)SIN(B)} + D_{F_1,U_1} \text{SIN}^2\text{(A)COS}^2\text{(B)}$$

$$+ D_{F_3,U_1} \text{COS(A)SIN(A)COS(B)} + D_{F_1,U_3} \text{COS(A)SIN(A)COS(B)}$$

$$+ D_{F_3,U_3} \text{COS}^2\text{(A))DEL}(U_3) + (-D_{F_2,U_1} \text{SIN(A)SIN}^2\text{(B)}$$

$$- D_{F_2,U_2} \text{SIN(A)COS(B)SIN(B)} + D_{F_1,U_1} \text{SIN(A)COS(B)SIN(B)}$$

$$+ D_{F_3,U_1} \text{COS(A)SIN(B)} + D_{F_1,U_2} \text{SIN(A)COS}^2\text{(B)}$$

$$+ D_{F_3,U_2} \text{COS(A)COS(B))DEL}(U_2)$$

$$+ (D_{F_2,U_2} \text{COS(A)SIN(A)SIN}^2\text{(B)}$$

$$- D_{F_2,U_1} \text{COS(A)SIN(A)COS(B)SIN(B)}$$

$$- D_{F_1,U_2} \text{COS(A)SIN(A)COS(B)SIN(B)} + D_{F_2,U_3} \text{SIN}^2\text{(A)SIN(B)}$$

$$- D_{F_3,U_2} \text{COS}^2\text{(A)SIN(B)} + D_{F_1,U_1} \text{COS(A)SIN(A)COS}^2\text{(B)}$$

$$- D_{F_1,U_3} \text{SIN}^2\text{(A)COS(B)} + D_{F_3,U_1} \text{COS}^2\text{(A)COS(B)}$$

$$- D_{F_3,U_3} \text{COS(A)SIN(A)) DEL}(U_1)$$

## Forces Produced by Angular Velocity Perturbations

The program used in the preceding section can, with suitable notational changes, be used to formulate the forces produced by angular velocity perturbations. However, whereas in the preceding application the required forces were obtained by multiplying the transformed aerodynamic stability derivatives by linear velocity increments, in the present case the transformed derivatives must be multiplied by angular velocity increments. In view of these similarities, the following program and displayed forces will be presented without further comment, except to point out that the aerodynamic stability derivatives of the $i$th force with respect to the $j$th angular velocity component are denoted by $D_{F_i,P_j}$. The corresponding transformed derivatives are denoted by $TD_{F_i,P_j}$, and the resulting forces by $FDP_i$.

```
(C20) TDP[I,J]:=0$

(C21) FOR I:1 THRU 3 DO FOR J:1 THRU 3 DO
 FOR M:1 THRU 3 DO FOR N:1 THRU 3 DO
 TDP[I,J]:TDP[I,J]+DIFF(Y[I],X[M])*DIFF(Y[J],X[N])*D[F[M],P[N]]$

(C22) FDP[I]:=0$

(C23) FOR I:1 THRU 3 DO FOR J:1 THRU 3 DO
 FDP[I]:FDP[I]+TDP[I,J]*DEL(P[J])$

(C24) FOR I:1 THRU 3 DO DISPLAY(FDP[I]);
```

$$FDP_1 = (D_{F_2,P_2}\ \mathrm{COS(A)SIN(A)SIN^2(B)}$$

$$- D_{F_2,P_1}\ \mathrm{COS(A)SIN(A)COS(B)SIN(B)}$$

$$- D_{F_1,P_2}\ \mathrm{COS(A)SIN(A)COS(B)SIN(B)} + D_{F_3,P_2}\ \mathrm{SIN^2(A)SIN(B)}$$

$$- D_{F_2,P_3}\ \mathrm{COS^2(A)SIN(B)} + D_{F_1,P_1}\ \mathrm{COS(A)SIN(A)COS^2(B)}$$

$$- D_{F_3,P_1}\ \mathrm{SIN^2(A)COS(B)} + D_{F_1,P_3}\ \mathrm{COS^2(A)COS(B)}$$

$$- D_{F_3,P_3}\ \mathrm{COS(A)SIN(A))DEL(P_3)}$$

$$+ (-D_{F_2,P_1}\ \mathrm{COS(A)SIN^2(B)} - D_{F_2,P_2}\ \mathrm{COS(A)COS(B)SIN(B)}$$

$$+ D_{F_1,P_1}\ \mathrm{COS(A)COS(B)SIN(B)} - D_{F_3,P_1}\ \mathrm{SIN(A)SIN(B)}$$

$$+ D_{F_1,P_2}\ \mathrm{COS(A)COS^2(B)} - D_{F_3,P_2}\ \mathrm{SIN(A)COS(B))DEL(P_2)}$$

$$+ (D_{F_2,P_2} \cos^2(A)\sin^2(B) - D_{F_2,P_1} \cos^2(A)\cos(B)\sin(B)$$

$$- D_{F_1,P_2} \cos^2(A)\cos(B)\sin(B) + D_{F_3,P_2} \cos(A)\sin(A)\sin(B)$$

$$+ D_{F_2,P_3} \cos(A)\sin(A)\sin(B) + D_{F_1,P_1} \cos^2(A)\cos^2(B)$$

$$- D_{F_3,P_1} \cos(A)\sin(A)\cos(B) - D_{F_1,P_3} \cos(A)\sin(A)\cos(B)$$

$$+ D_{F_3,P_3} \sin^2(A))\mathrm{DEL}(P_1)$$

$$\mathrm{FDP}_2 = (-D_{F_1,P_2} \sin(A)\sin^2(B) - D_{F_2,P_2} \sin(A)\cos(B)\sin(B)$$

$$+ D_{F_1,P_1} \sin(A)\cos(B)\sin(B) + D_{F_1,P_3} \cos(A)\sin(B)$$

$$+ D_{F_2,P_1} \sin(A)\cos^2(B) + D_{F_2,P_3} \cos(A)\cos(B))\mathrm{DEL}(P_3)$$

$$+ D_{F_1,P_1} \sin^2(B) + D_{F_2,P_1} \cos(B)\sin(B) + D_{F_1,P_2} \cos(B)\sin(B)$$

$$+ D_{F_2,P_2} \cos^2(B))\mathrm{DEL}(P_2) + (-D_{F_1,P_2} \cos(A)\sin^2(B)$$

$$- D_{F_2,P_2} \cos(A)\cos(B)\sin(B) + D_{F_1,P_1} \cos(A)\cos(B)\sin(B)$$

$$- D_{F_1,P_3} \sin(A)\sin(B) + D_{F_2,P_1} \cos(A)\cos^2(B)$$

$$- D_{F_2,P_3} \sin(A)\cos(B))\mathrm{DEL}(P_1)$$

$$\mathrm{FDP}_3 = (D_{F_2,P_2} \sin^2(A)\sin^2(B) - D_{F_2,P_1} \sin^2(A)\cos(B)\sin(B)$$

$$- D_{F_1,P_2} \sin^2(A)\cos(B)\sin(B) - D_{F_3,P_2} \cos(A)\sin(A)\sin(B)$$

$$- D_{F_2,P_3} \cos(A)\sin(A)\sin(B) + D_{F_1,P_1} \sin^2(A)\cos^2(B)$$

$$+ D_{F_3,P_1} \cos(A)\sin(A)\cos(B) + D_{F_1,P_3} \cos(A)\sin(A)\cos(B)$$

$$+ D_{F_3,P_3}\ \mathrm{COS}^2(\mathrm{A}))\mathrm{DEL}(P_3) + (-D_{F_2,P_1}\ \mathrm{SIN}(\mathrm{A})\mathrm{SIN}^2(\mathrm{B})$$

$$- D_{F_2,P_2}\ \mathrm{SIN}(\mathrm{A})\mathrm{COS}(\mathrm{B})\mathrm{SIN}(\mathrm{B}) + D_{F_1,P_1}\ \mathrm{SIN}(\mathrm{A})\mathrm{COS}(\mathrm{B})\mathrm{SIN}(\mathrm{B})$$

$$+ D_{F_3,P_1}\ \mathrm{COS}(\mathrm{A})\mathrm{SIN}(\mathrm{B}) + D_{F_1,P_2}\ \mathrm{SIN}(\mathrm{A})\mathrm{COS}^2(\mathrm{B})$$

$$+ D_{F_3,P_2}\ \mathrm{COS}(\mathrm{A})\mathrm{COS}(\mathrm{B}))\mathrm{DEL}(P_2)$$

$$+ (D_{F_2,P_2}\ \mathrm{COS}(\mathrm{A})\mathrm{SIN}(\mathrm{A})\mathrm{SIN}^2(\mathrm{B})$$

$$- D_{F_2,P_1}\ \mathrm{COS}(\mathrm{A})\mathrm{SIN}(\mathrm{A})\mathrm{COS}(\mathrm{B})\mathrm{SIN}(\mathrm{B})$$

$$- D_{F_1,P_2}\ \mathrm{COS}(\mathrm{A})\mathrm{SIN}(\mathrm{A})\mathrm{COS}(\mathrm{B})\mathrm{SIN}(\mathrm{B})$$

$$+ D_{F_2,P_3}\ \mathrm{SIN}^2(\mathrm{A})\mathrm{SIN}(\mathrm{B}) - D_{F_3,P_2}\ \mathrm{COS}^2(\mathrm{A})\mathrm{SIN}(\mathrm{B})$$

$$+ D_{F_1,P_1}\ \mathrm{COS}(\mathrm{A})\mathrm{SIN}(\mathrm{A})\mathrm{COS}^2(\mathrm{B}) - D_{F_1,P_3}\ \mathrm{SIN}^2(\mathrm{A})\mathrm{COS}(\mathrm{B})$$

$$+ D_{F_3,P_1}\ \mathrm{COS}^2(\mathrm{A})\mathrm{COS}(\mathrm{B}) - D_{F_3,P_3}\ \mathrm{COS}(\mathrm{A})\mathrm{SIN}(\mathrm{A}))\mathrm{DEL}(P_1)$$

## Forces Produced by Linear Acceleration Perturbations

The procedure used in the preceding two sections may, with equal facility, be used to formulate the aerodynamic forces produced by linear acceleration perturbations. However, in this case the required forces are obtained by multiplying the transformed aerodynamic stability derivatives, with respect to acceleration components, by linear acceleration increments. The aerodynamic stability derivatives of the $i$th force component $F_i$ with respect to the $j$th linear acceleration component $A_j$ are denoted by $DF_{j,A_j}$, and the transformed derivatives by $TDF_{i,A_j}$. The corresponding force components in body axes are denoted by $FDA_i$.

Due to the fact that lift responds in a transient manner when, for example, the angle of attack $A$ or the linear velocity component $U_3$ is suddenly changed, the acceleration derivatives are very different from the velocity derivatives, which can be determined on the basis of steady-state aerodynamics. This is a consequence of the fact that the pressure distribution on a wing or tail surface does not adjust itself instantaneously to its equilibrium value when the angle of attack or the velocity components are suddenly changed. Hence, in order to get a sufficiently accurate description of these derivatives during the indicial response phase, it may be necessary to use function generation or look-up tables (ref. 5).

When the program of the preceding section has been modified to incorporate the necessary notational changes, it assumes the following form:

```
(C25) TDA[I,J]:=0$

(C26) FOR I:1 THRU 3 DO FOR J:1 THRU 3 DO
 FOR M:1 THRU 3 DO FOR N:1 THRU 3 DO
 TDA[I,J]:TDA[I,J]+DIFF(Y[I],X[M])*DIFF(Y[J],X[N])*D[F[M],A[N]]$

(C27) FDA[I]:=0$

(C28) FOR I:1 THRU 3 DO FOR J:1 THRU 3 DO
 FDA[I]:FDA[I]+TDA[I,J]*DEL(A[J])$

(C29) FOR I:1 THRU 3 DO DISPLAY(FDA[I])$
```

Execution of this program yields the aerodynamic forces produced by linear acceleration perturbations. These are

$$FDA_1 = (D_{F_2,A_2} \cos(A)\sin(A)\sin^2(B)$$

$$- D_{F_2,A_1} \cos(A)\sin(A)\cos(B)\sin(B)$$

$$- D_{F_1,A_2} \cos(A)\sin(A)\cos(B)\sin(B) + D_{F_3,A_2} \sin^2(A)\sin(B)$$

$$- D_{F_2,A_3} \cos^2(A)\sin(B) + D_{F_1,A_1} \cos(A)\sin(A)\cos^2(B)$$

$$- D_{F_3,A_1} \sin^2(A)\cos(B) + D_{F_1,A_3} \cos^2(A)\cos(B)$$

$$- D_{F_3,A_3} \cos(A)\sin(A)DEL(A_3)$$

$$+ (-D_{F_2,A_1} \cos(A)\sin^2(B) - D_{F_2,A_2} \cos(A)\cos(B)\sin(B)$$

$$+ D_{F_1,A_1} \cos(A)\cos(B)\sin(B) - D_{F_3,A_1} \sin(A)\sin(B)$$

$$+ D_{F_1,A_2} \cos(A)\cos^2(B) - D_{F_3,A_2} \sin(A)\cos(B))DEL(A_2)$$

$$+ (D_{F_2,A_2} \cos^2(A)\sin^2(B) - D_{F_2,A_1} \cos^2(A)\cos(B)\sin(B)$$

$$- D_{F_1,A_2} \cos^2(A)\cos(B)\sin(B) + D_{F_3,A_2} \cos(A)\sin(A)\sin(B)$$

$$+ D_{F_2,A_3} \cos(A)\sin(A)\sin(B) + D_{F_1,A_1} \cos^2(A)\cos^2(B)$$

$$- D_{F_3,A_1} \cos(A)\sin(A)\cos(B) - D_{F_1,A_3} \cos(A)\sin(A)\cos(B)$$

$$+ D_{F_3,A_3} \sin^2(A))\mathrm{DEL}(A_1)$$

$$\mathrm{FDA}_2 = (-D_{F_1,A_2} \sin(A)\sin^2(B) - D_{F_2,A_2} \sin(A)\cos(B)\sin(B)$$

$$+ D_{F_1,A_1} \sin(A)\cos(B)\sin(B) + D_{F_1,A_3} \cos(A)\sin(B)$$

$$+ D_{F_2,A_1} \sin(A)\cos^2(B) + D_{F_2,A_3} \cos(A)\cos(B))\mathrm{DEL}(A_3)$$

$$+ (D_{F_1,A_1} \sin^2(B) + D_{F_2,A_1} \cos(B)\sin(B) + D_{F_1,A_2} \cos(B)\sin(B)$$

$$+ D_{F_2,A_2} \cos^2(B))\mathrm{DEL}(A_2) + (-D_{F_1,A_2} \cos(A)\sin^2(B)$$

$$- D_{F_2,A_2} \cos(A)\cos(B)\sin(B) + D_{F_1,A_1} \cos(A)\cos(B)\sin(B)$$

$$- D_{F_1,A_3} \sin(A)\sin(B) + D_{F_2,A_1} \cos(A)\cos^2(B)$$

$$- D_{F_2,A_3} \sin(A)\cos(B)\mathrm{DEL}(A_1)$$

$$\mathrm{FDA}_3 = (D_{F_2,A_2} \sin^2(A)\sin^2(B) - D_{F_2,A_1} \sin^2(A)\cos(B)\sin(B)$$

$$- D_{F_1,A_2} \sin^2(A)\cos(B)\sin(B) - D_{F_3,A_2} \cos(A)\sin(A)\sin(B)$$

$$- D_{F_2,A_3} \cos(A)\sin(A)\sin(B) + D_{F_1,A_1} \sin^2(A)\cos^2(B)$$

$$+ D_{F_3,A_1} \cos(A)\sin(A)\cos(B) + D_{F_1,A_3} \cos(A)\sin(A)\cos(B)$$

$$+ D_{F_3,A_3} \cos^2(A))\mathrm{DEL}(A_3) + (-D_{F_2,A_1} \sin(A)\sin^2(B)$$

$$- D_{F_2,A_2} \sin(A)\cos(B)\sin(B) + D_{F_1,A_1} \sin(A)\cos(B)\sin(B)$$

$$+ D_{F_3,A_1} \mathrm{COS(A)SIN(B)} + D_{F_1,A_2} \mathrm{SIN(A)COS^2(B)}$$

$$+ D_{F_3,A_2} \mathrm{COS(A)COS(B))DEL(A_2)}$$

$$+ (D_{F_2,A_2} \mathrm{COS(A)SIN(A)SIN^2(B)}$$

$$- D_{F_2,A_1} \mathrm{COS(A)SIN(A)COS(B)SIN(B)}$$

$$- D_{F_1,A_2} \mathrm{COS(A)SIN(A)COS(B)SIN(B)}$$

$$+ D_{F_2,A_3} \mathrm{SIN^2(A)SIN(B)} - D_{F_3,A_2} \mathrm{COS^2(A)SIN(B)}$$

$$+ D_{F_1,A_1} \mathrm{COS(A)SIN(A)COS^2(B)} - D_{F_1,A_3} \mathrm{SIN^2(A)COS(B)}$$

$$+ D_{F_3,A_1} \mathrm{COS^2(A)COS(B)} - D_{F_3,A_3} \mathrm{COS(A)SIN(A))DEL(A_1)}$$

The components of the resultant aerodynamic force are

```
(C30) FOR I:1 THRU 3 DO FA[I]:FDU[I]+FDP[I]+FDA[I]+CF[I]+SF[I]$
```

## Gravity Forces

The gravitational force vector acting on an aircraft has the value $M\bar{g}$, where $M$ is the mass of the aircraft and $\bar{g}$ is the gravitational acceleration vector. The magnitude of $\bar{g}$ is assumed constant, which is tantamount to the assumption of a flat earth. The gravity vector is specified in an earth-fixed reference frame; and it is required to find the components of this vector in aircraft body axes. In accordance with aeronautical convention, a transformation from earth-fixed axes to aircraft body axes involves a rotation $R_3$ about the $Y_3$ body axis, followed by a rotation $R_2$ about the $Y_2$ body axis, and a rotation $R_1$ about the $Y_1$ body axis. Hence, if it is assumed that the body axes and the earth-fixed axes are initially coincident, the components of the gravitational force $FG_i$ in body axes are given by an equation of the form

$$[FG] = [R_1][R_2][R_3][Mg]$$

where $[FG]$ is a column vector of body axes components, $[R_1]$, $[R_2]$, and $[R_3]$ are rotation matrices, and $[Mg]$ is a column vector of earth-fixed axes components. These matrix operations can be performed by MACSYMA to yield the required force components in body axes as follows:

```
(C31) ENTERMATRIX(3,3);

ROW 1 COLUMN 1  COS(R[3]);

ROW 1 COLUMN 2  SIN(R[3]);

ROW 1 COLUMN 3  0;

ROW 2 COLUMN 1  -SIN([3]);

ROW 2 COLUMN 2  COS(R[3]);

ROW 2 COLUMN 3  0;

ROW 3 COLUMN 1  0;

ROW 3 COLUMN 2  0;

ROW 3 COLUMN 3  1;

MATRIX-ENTERED
```

(D31)
$$\begin{bmatrix} \cos(R_3) & \sin(R_3) & 0 \\ -\sin(R_3) & \cos(R_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
(C32) ENTERMATRIX(3,3);

ROW 1 COLUMN 1  COS(R[2]);

ROW 1 COLUMN 2  0;

ROW 1 COLUMN 3  -SIN(R[2]);

ROW 2 COLUMN 1  0;

ROW 2 COLUMN 2  1;

ROW 2 COLUMN 3  0;

ROW 3 COLUMN 1  SIN(R[2]);

ROW 3 COLUMN 2  0;

ROW 3 COLUMN 3  COS(R[2]);
```

```
MATRIX-ENTERED
                          [ COS(R )   0   -SIN(R )  ]
                          [      2              2   ]
(D32)                     [    0      1      0      ]
                          [                         ]
                          [ SIN(R )   0   COS(R )   ]
                          [      2              2   ]

(C33) ENTERMATRIX(3,3);

ROW 1 COLUMN 1  1;

ROW 1 COLUMN 2  0;

ROW 1 COLUMN 3  0;

ROW 2 COLUMN 1  0;

ROW 2 COLUMN 2  COS(R[1]);

ROW 2 COLUMN 3  SIN(R[1]);

ROW 3 COLUMN 1  0;

ROW 3 COLUMN 2  -SIN(R[1]);

ROW 3 COLUMN 3  COS(R[1]);

MATRIX-ENTERED
                          [ 1      0          0     ]
                          [                         ]
(D33)                     [ 0   COS(R )    SIN(R )  ]
                          [          1          1   ]
                          [ 0  -SIN(R )    COS(R )  ]
                          [          1          1   ]

(C34) ENTERMATRIX(3,1);

ROW 1 COLUMN 1  0;

ROW 2 COLUMN 1  0;

ROW 3 COLUMN 1  M*G;

MATRIX-ENTERED                      [  0  ]
                                    [     ]
(D34)                               [  0  ]
                                    [     ]
                                    [ G M ]
```

The product of these four matrices gives the following column vector of gravitational force components relative to aircraft body axes:

```
(C35) (D33).(D32).(D31).(D34);
```

$$(D35) \qquad \begin{bmatrix} -\mathrm{SIN}(R_2)\ G\ M \\ \mathrm{SIN}(R_1)\ \mathrm{COS}(R_2)\ G\ M \\ \mathrm{COS}(R_1)\ \mathrm{COS}(R_2)\ G\ M \end{bmatrix}$$

These vector components may be expressed in conventional form by executing the following two programming steps, which yield:

```
(C36) FOR I:1 THRU 3 DO ROW[I]:FIRST(ROW((D35),I))$

(C37) FOR I:1 THRU 3 DO (FG[I]:ROW[I][1],DISPLAY(FG[I]))$
```

$$FG_1 = -\mathrm{SIN}(R_2)\ G\ M$$

$$FG_2 = \mathrm{SIN}(R_1)\ \mathrm{COS}(R_2)\ G\ M$$

$$FG_3 = \mathrm{COS}(R_1)\ \mathrm{COS}(R_2)\ G\ M$$

where $R_i = (R_i^O + \delta R_i)$, $R_i^O$ are equilibrium values, and $\delta R_i$ are angular perturbations.

## Inertia Forces

The formulation of the inertia forces involves the determination of the product of an angular velocity matrix and a column vector of linear velocity components. This product is the matrix equivalent of the familiar vector product $\bar{\omega} \times \bar{V}$. By adding to the components of this vector, the components of linear acceleration relative to aircraft body axes, the components of inertial acceleration relative to these axes are obtained. The required matrices may be entered and multiplied as follows:

```
(C38) ENTERMATRIX(3,3);

ROW 1 COLUMN 1  0;

ROW 1 COLUMN 2  -P[3];

ROW 1 COLUMN 3  P[2];

ROW 2 COLUMN 1  P[3];
```

```
ROW 2 COLUMN 2  0;

ROW 2 COLUMN 3  -P[1];

ROW 3 COLUMN 1  -P[2];

ROW 3 COLUMN 2  P[1];

ROW 3 COLUMN 3  0;

MATRIX-ENTERED
```

(D38)
$$\begin{bmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}$$

```
(C39) ENTERMATRIX(3,1);

ROW 1 COLUMN 1  U[1];

ROW 2 COLUMN 1  U[2];

ROW 3 COLUMN 1  U[3];

MATRIX-ENTERED
```

(D39)
$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

```
(C40) (D38).(D39);
```

(D40)
$$\begin{bmatrix} P_2 U_3 - U_2 P_3 \\ U_1 P_3 - P_1 U_3 \\ P_1 U_2 - U_1 P_2 \end{bmatrix}$$

```
(C41) FOR I:1 THRU 3 DO ROW[I]:FIRST(ROW((D40),I))$

(C42) FOR I:1 THRU 3 DO (C[I]:ROW[I][1],DISPLAY(C[I]))$
```

$$C_1 = P_2\ U_3 - U_2\ P_3$$

$$C_2 = U_1\ P_3 - P_1\ U_3$$

$$C_3 = P_1\ U_2 - U_1\ P_2$$

A statement of the fact that the $i$th component of the linear velocity vector is a function of time, requires the use of the DEPENDENCIES function. The use of this function permits the system to differentiate the components $U_i$ with respect to time. The remaining two programming statements request the system to add the components, multiply the individual sums by the mass $M$ of the vehicle, and display the resulting inertial force components $FR_i$ as follows:

```
(C43) DEPENDENCIES(U(I,T))$
(C44) FOR I:1 THRU 3 DO FR[I]:M*(C[I]+DIFF(U[I],T))$
(C45) FOR I:1 THRU 3 DO DISPLAY(FR[I])$
```

$$FR_1 = \left(\frac{d}{dT} U_1 + P_2\ U_3 - U_2\ P_3\right) M$$

$$FR_2 = \left(\frac{d}{dT} U_2 - P_1\ U_3 + U_1\ P_3\right) M$$

$$FR_3 = M\ \left(-U_1\ P_2 + \frac{d}{dT} U_3 + P_1\ U_2\right)$$

Resultant Forces

It only remains to request MACSYMA to combine the aerodynamic, gravitational, and inertia forces that were formulated in preceding sections and display the results. The $i$th component of the resultant force will be denoted by $FT_i$ where $T_i$ is the $i$th component of thrust. The two programming steps and the formulated equations follow.

```
(C46) FOR I:1 THRU 3 DO FT[I]:FR[I]-FG[I]-FA[I]$
(C47) FOR I:1 THRU 3 DO DISPLAY(FT[I])$
```

$$FT_1 = -(-D_{F_2,C_k}\ COS(A)SIN(B) + D_{F_1,C_K}\ COS(A)COS(B)$$

$$- D_{F_3,C_K}\ SIN(A))DEL(C_K) - (D_{F_2,U_2}\ COS(A)SIN(A)SIN^2(B)$$

$$- D_{F_2,U_1}\ COS(A)SIN(A)COS(B)SIN(B)$$

$$- D_{F_1,U_2} COS(A)SIN(A)COS(B)SIN(B) + D_{F_3,U_2} SIN^2(A)SIN(B)$$

$$- D_{F_2,U_3} COS^2(A)SIN(B) + D_{F_1,U_1} COS(A)SIN(A)COS^2(B)$$

$$- D_{F_3,U_1} SIN^2(A)COS(B) + D_{F_1,U_3} COS^2(A)COS(B)$$

$$- D_{F_3,U_3} COS(A)SIN(A))DEL(U_3) - (D_{F_2,P_2} COS(A)SIN(A)SIN^2(B)$$

$$- D_{F_2,P_1} COS(A)SIN(A)COS(B)SIN(B)$$

$$- D_{F_1,P_2} COS(A)SIN(A)COS(B)SIN(B) + D_{F_3,P_2} SIN^2(A)SIN(B)$$

$$- D_{F_2,P_3} COS^2(A)SIN(B) + D_{F_1,P_1} COS(A)SIN(A)COS^2(B)$$

$$- D_{F_3,P_1} SIN^2(A)COS(B) + D_{F_1,P_3} COS^2(A)COS(B)$$

$$- D_{F_3,P_3} COS(A)SIN(A))DEL(P_3) - (D_{F_2,A_2} COS(A)SIN(A)SIN^2(B)$$

$$- D_{F_2,A_1} COS(A)SIN(A)COS(B)SIN(B)$$

$$- D_{F_1,A_2} COS(A)SIN(A)COS(B)SIN(B) + D_{F_3,A_2} SIN^2(A)SIN(B)$$

$$- D_{F_2,A_3} COS^2(A)SIN(B) + D_{F_1,A_1} COS(A)SIN(A)COS^2(B)$$

$$- D_{F_3,A_1} SIN^2(A)COS(B) + D_{F_1,A_3} COS^2(A)COS(B)$$

$$- D_{F_3,A_3} COS(A)SIN(A))DEL(A_3) - (-D_{F_2,U_1} COS(A)SIN^2(B)$$

$$- D_{F_2,U_2} COS(A)COS(B)SIN(B) + D_{F_1,U_1} COS(A)COS(B)SIN(B)$$

$$- D_{F_3,U_1} SIN(A)SIN(B) + D_{F_1,U_2} COS(A)COS^2(B)$$

$$- D_{F_3,U_2} SIN(A)COS(B))DEL(U_2) - (-D_{F_2,P_1} COS(A)SIN^2(B)$$

$$- D_{F_2,P_2} \, COS(A)COS(B)SIN(B) + D_{F_1,P_1} \, COS(A)COS(B)SIN(B)$$

$$- D_{F_3,P_1} \, SIN(A)SIN(B) - D_{F_1,P_2} \, COS(A)COS^2(B)$$

$$- D_{F_3,P_2} \, SIN(A)COS(B))DEL(P_2) - (-D_{F_2,A_1} \, COS(A)SIN^2(B)$$

$$- D_{F_2,A_2} \, COS(A)COS(B)SIN(B) - D_{F_1,A_1} \, COS(A)COS(B)SIN(B)$$

$$- D_{F_3,A_1} \, SIN(A)SIN(B) + D_{F_1,A_2} \, COS(A)COS^2(B)$$

$$- D_{F_3,A_2} \, SIN(A)COS(B))DEL(A_2) - (D_{F_2,U_2} \, COS^2(A)SIN^2(B)$$

$$- D_{F_2,U_1} \, COS^2(A)COS(B)SIN(B) - D_{F_1,U_2} \, COS^2(A)COS(B)SIN(B)$$

$$+ D_{F_3,U_2} \, COS(A)SIN(A)SIN(B) + D_{F_2,U_3} \, COS(A)SIN(A)SIN(B)$$

$$+ D_{F_1,U_1} \, COS^2(A)COS^2(B) - D_{F_3,U_1} \, COS(A)SIN(A)COS(B)$$

$$- D_{F_1,U_3} \, COS(A)SIN(A)COS(B) + D_{F_3,U_3} \, SIN^2(A))DEL(U_1)$$

$$- (D_{F_2,P_2} \, COS^2(A)SIN^2(B) - D_{F_2,P_1} \, COS^2(A)COS(B)SIN(B)$$

$$- D_{F_1,P_2} \, COS^2(A)COS(B)SIN(B) + D_{F_3,P_2} \, COS(A)SIN(A)SIN(B)$$

$$+ D_{F_2,P_3} \, COS(A)SIN(A)SIN(B) + D_{F_1,P_1} \, COS^2(A)COS^2(B)$$

$$- D_{F_3,P_1} \, COS(A)SIN(A)COS(B) - D_{F_1,P_3} \, COS(A)SIN(A)COS(B)$$

$$+ D_{F_3,P_3} \, SIN^2(A))DEL(P_1) - (D_{F_2,A_2} \, COS^2(A)SIN^2(B)$$

$$- D_{F_2,A_1} \, COS^2(A)COS(B)SIN(B) - D_{F_1,A_2} \, COS^2(A)COS(B)SIN(B)$$

$$+ D_{F_3,A_2} \, COS(A)SIN(A)SIN(B) + D_{F_2,A_3} \, COS(A)SIN(A)SIN(B)$$

$$+ D_{F_1,A_1} \cos^2(A)\cos^2(B) - D_{F_3,A_1} \cos(A)\sin(A)\cos(B)$$

$$- D_{F_1,A_3} \cos(A)\sin(A)\cos(B) + D_{F_3,A_3} \sin^2(A))DEL(A_1)$$

$$+ \sin(R_2)GM + (\frac{d}{dT} U_1 + P_2U_3 - U_2P_3)M$$

$$+ S_{F_2} \cos(A)\sin(B) - S_{F_1} \cos(A)\cos(B) + S_{F_3} \sin(A)$$

$$FT_2 = -(D_{F_1,C_K} \sin(B) + D_{F_2,C_K} \cos(B))DEL(C_K)$$

$$- (-D_{F_1,U_2} \sin(A)\sin^2(B) - D_{F_2,U_2} \sin(A)\cos(B)\sin(B)$$

$$+ D_{F_1,U_1} \sin(A)\cos(B)\sin(B) + D_{F_1,U_3} \cos(A)\sin(B)$$

$$+ D_{F_2,U_1} \sin(A)\cos^2(B) + D_{F_2,U_3} \cos(A)\cos(B))DEL(U_3)$$

$$- (-D_{F_1,P_2} \sin(A)\sin^2(B) - D_{F_2,P_2} \sin(A)\cos(B)\sin(B)$$

$$+ D_{F_1,P_1} \sin(A)\cos(B)\sin(B) + D_{F_1,P_3} \cos(A)\sin(B)$$

$$+ D_{F_2,P_1} \sin(A)\cos^2(B) + D_{F_2,P_3} \cos(A)\cos(B))DEL(P_3)$$

$$- (-D_{F_1,A_2} \sin(A)\sin^2(B) - D_{F_2,A_2} \sin(A)\cos(B)\sin(B)$$

$$+ D_{F_1,A_1} \sin(A)\cos(B)\sin(B) + D_{F_1,A_3} \cos(A)\sin(B)$$

$$+ D_{F_2,A_1} \sin(A)\cos^2(B) + D_{F_2,A_3} \cos(A)\cos(B))DEL(A_3)$$

$$- (D_{F_1,U_1} \sin^2(B) + D_{F_2,U_1} \cos(B)\sin(B) + D_{F_1,U_2} \cos(B)\sin(B)$$

$$+ D_{F_2,U_2} \cos^2(B))DEL(U_2) - (D_{F_1,P_1} \sin^2(B) + D_{F_2,P_1} \cos(B)\sin(B)$$

$$+ D_{F_1,P_2} \cos(B)\sin(B) + D_{F_2,P_2} \cos^2(B))DEL(P_2) - (D_{F_1,A_1} \sin^2(B)$$

$$+ D_{F_2,A_1} \cos(B)\sin(B) + D_{F_1,A_2} \cos(B)\sin(B)$$

$$+ D_{F_2,A_2} \cos^2(B))DEL(A_2) - (-D_{F_1,U_2} \cos(A)\sin^2(B)$$

$$- D_{F_2,U_2} \cos(A)\cos(B)\sin(B) + D_{F_1,U_1} \cos(A)\cos(B)\sin(B)$$

$$- D_{F_1,U_3} \sin(A)\sin(B) + D_{F_2,U_1} \cos(A)\cos^2(B)$$

$$- D_{F_2,U_3} \sin(A)\cos(B))DEL(U_1) - (-D_{F_1,P_2} \cos(A)\sin^2(B)$$

$$- D_{F_2,P_2} \cos(A)\cos(B)\sin(B) + D_{F_1,P_1} \cos(A)\cos(B)\sin(B)$$

$$- D_{F_1,P_3} \sin(A)\sin(B) + D_{F_2,P_1} \cos(A)\cos^2(B)$$

$$- D_{F_2,P_3} \sin(A)\cos(B))DEL(P_1) - (-D_{F_1,A_2} \cos(A)\sin^2(B)$$

$$- D_{F_2,A_2} \cos(A)\cos(B)\sin(B) + D_{F_1,A_1} \cos(A)\cos(B)\sin(B)$$

$$- D_{F_1,A_3} \sin(A)\sin(B) + D_{F_2,A_1} \cos(A)\cos^2(B)$$

$$- D_{F_2,A_3} \sin(A)\cos(B))DEL(A_1) - \sin(R_1)\cos(R_2)GM$$

$$+ (\frac{d}{dT} U_2 - P_1U_3 + U_1P_3)M - S_{F_1} \sin(B) - S_{F_2} \cos(B)$$

$$FT_3 = -(-D_{F_2,C_K} \sin(A)\sin(B) + D_{F_1,C_K} \sin(A)\cos(B)$$

$$+ D_{F_3,C_K} \cos(A))DEL(C_K) - (D_{F_2,U_2} \sin^2(A)\sin^2(B)$$

$$- D_{F_2,U_1} \sin^2(A)\cos(B)\sin(B) - D_{F_1,U_2} \sin^2(A)\cos(B)\sin(B)$$

$$- D_{F_3,U_2} \cos(A)\sin(A)\sin(B) - D_{F_2,U_3} \cos(A)\sin(A)\sin(B)$$

$$+ D_{F_1,U_1} \sin^2(A)\cos^2(B) + D_{F_3,U_1} \cos(A)\sin(A)\cos(B)$$

$$+ D_{F_1,U_3} COS(A)SIN(A)COS(B) + D_{F_3,U_3} COS^2(A))DEL(U_3)$$

$$- (D_{F_2,P_2} SIN^2(A)SIN^2(B) - D_{F_2,P_1} SIN^2(A)COS(B)SIN(B)$$

$$- D_{F_1,P_2} SIN^2(A)COS(B)SIN(B) - D_{F_3,P_2} COS(A)SIN(A)SIN(B)$$

$$- D_{F_2,P_3} COS(A)SIN(A)SIN(B) + D_{F_1,P_1} SIN^2(A)COS^2(B)$$

$$+ D_{F_3,P_1} COS(A)SIN(A)COS(B) + D_{F_1,P_1} COS(A)SIN(A)COS^2(B)$$

$$- D_{F_1,P_3} SIN^2(A)COS(B) + D_{F_3,P_1} COS^2(A)COS(B)$$

$$- D_{F_3,P_3} COS(A)SIN(A))DEL(P_1) - (D_{F_2,A_2} COS(A)SIN(A)SIN^2(B)$$

$$- D_{F_2,A_1} COS(A)SIN(A)COS(B)SIN(B)$$

$$- D_{F_1,A_2} COS(A)SIN(A)COS(B)SIN(B) + D_{F_2,A_3} SIN^2(A)SIN(B)$$

$$- D_{F_3,A_2} COS^2(A)SIN(B) + D_{F_1,A_1} COS(A)SIN(A)COS^2(B)$$

$$- D_{F_1,A_3} SIN^2(A)COS(B) + D_{F_3,A_1} COS^2(A)COS(B)$$

$$- D_{F_3,A_3} COS(A)SIN(A))DEL(A_1) + M(-U_1P_2 + \frac{d}{dT} U_3$$

$$+ P_1U_2) - COS(R_1)COS(R_2)GM + S_{F_2} SIN(A)SIN(B)$$

$$- S_{F_1} SIN(A)COS(B) - S_{F_3} COS(A)$$

$$+ D_{F_1,P_2} SIN(A)\ COS^2(B) + D_{F_3,P_2} COS(A)\ COS(B))\ DEL(P_2)$$

$$- (-D_{F_2,A_1} SIN(A)\ SIN^2(B) - D_{F_2,A_2} SIN(A)\ COS(B)\ SIN(B)$$

$$+ D_{F_1,A_1} SIN(A)\ COS(B)\ SIN(B) + D_{F_3,A_1} COS(A)\ SIN(B)$$

$$+ D_{F_1,A_2} \; SIN(A) \; COS^2(B) + D_{F_3,A_2} \; COS(A) \; COS(B)) \; DEL(A_2)$$

$$- (D_{F_2,U_2} \; COS(A) \; SIN(A) \; SIN^2(B) - D_{F_2,U_1} \; COS(A) \; SIN(A) \; COS(B) \; SIN(B)$$

$$- D_{F_1,U_2} \; COS(A) \; SIN(A) \; COS(B) \; SIN(B) + D_{F_2,U_3} \; SIN^2(A) \; SIN(B)$$

$$- D_{F_3,U_2} \; COS^2(A) \; SIN(B) + D_{F_1,U_1} \; COS(A) \; SIN(A) \; COS^2(B)$$

$$- D_{F_1,U_3} \; SIN^2(A) \; COS(B) + D_{F_3,U_1} \; COS^2(A) \; COS(B)$$

$$- D_{F_3,U_3} \; COS(A) \; SIN(A)) \; DEL(U_1) - (D_{F_2,P_2} \; COS(A) \; SIN(A) \; SIN^2(B)$$

$$- D_{F_2,P_1} \; COS(A) \; SIN(A) \; COS(B) \; SIN(B)$$

$$- D_{F_1,P_2} \; COS(A) \; SIN(A) \; COS(B) \; SIN(B)$$

$$+ D_{F_2,P_3} \; SIN^2(A) \; SIN(B) - D_{F_3,P_2} \; COS^2(A) \; SIN(B)$$

$$+ D_{F_1,P_3} \; COS(A)SIN(A)COS(B)$$

$$+ D_{F_3,P_3} \; COS^2(A))DEL(P_3) - (D_{F_2,A_2} \; SIN^2(A)SIN^2(B)$$

$$- D_{F_2,A_1} \; SIN^2(A)COS(B)SIN(B) - D_{F_1,A_2} \; SIN^2(A)COS(B)SIN(B)$$

$$- D_{F_3,A_2} \; COS(A)SIN(A)SIN(B) - D_{F_2,A_3} \; COS(A)SIN(A)SIN(B)$$

$$+ D_{F_1,A_1} \; SIN^2(A)COS^2(B) + D_{F_3,A_1} \; COS(A)SIN(A)COS(B)$$

$$+ D_{F_1,A_3} \; COS(A)SIN(A)COS(B) + D_{F_3,A_3} \; COS^2(A))DEL(A_3)$$

$$- (-D_{F_2,U_1} \; SIN(A)SIN^2(B) - D_{F_2,U_2} \; SIN(A)COS(B)SIN(B)$$

$$+ D_{F_1,U_1} \; SIN(A)COS(B)SIN(B) + D_{F_3,U_1} \; COS(A)SIN(B)$$

$$+ D_{F_1,U_2} \; SIN(A)COS^2(B) + D_{F_3,U_2} \; COS(A)COS(B))DEL(U_2)$$

$$- (-D_{F_2,P_1} \mathrm{SIN}(A)\mathrm{SIN}^2(B) - D_{F_2,P_2} \mathrm{SIN}(A)\mathrm{COS}(B)\mathrm{SIN}(B)$$

$$+ D_{F_1,P_1} \mathrm{SIN}(A)\mathrm{COS}(B)\mathrm{SIN}(B) + D_{F_3,P_1} \mathrm{COS}(A)\mathrm{SIN}(B)$$

## Special Forms of the Equations of Motion

In aeronautical studies involving small perturbations about the equilibrium or trim condition, the investigator sometimes wants to know how the vehicle will respond if the motion is restricted in some way. For example, he might wish to determine vehicle response in the absence of sideslip. MACSYMA is well equipped to implement assumptions of this type. By using a substitution command, MACSYMA goes through the equations, makes the required substitutions, and displays the modified results. For the case of zero sideslip the program requests MACSYMA to make the substitutions: SIN(B) = 0 and COS(B) = 1 in each force equation. The required substitution and display commands and the modified equations assume the following form:

```
(C48) FOR I:1 THRU 3 DO FT[I]:SUBST([SIN(B)=0,COS(B)=1],FT[I])$

(C49) FOR I:1 THRU 3 DO DISPLAY(FT[I])$
```

$$FT_1 = -(D_{F_1,C_K} \mathrm{COS}(A) - D_{F_3,C_K} \mathrm{SIN}(A))\mathrm{DEL}(C_K) - (-D_{F_3,U_1} \mathrm{SIN}^2(A)$$

$$- D_{F_3,U_3} \mathrm{COS}(A)\mathrm{SIN}(A) + D_{F_1,U_1} \mathrm{COS}(A)\mathrm{SIN}(A)$$

$$+ D_{F_1,U_3} \mathrm{COS}^2(A))\mathrm{DEL}(U_3) - (-D_{F_3,P_1} \mathrm{SIN}^2(A) - D_{F_3,P_3} \mathrm{COS}(A)\mathrm{SIN}(A)$$

$$+ D_{F_1,P_1} \mathrm{COS}(A)\mathrm{SIN}(A) + D_{F_1,P_3} \mathrm{COS}^2(A))\mathrm{DEL}(P_3) - (-D_{F_3,A_1} \mathrm{SIN}^2(A)$$

$$- D_{F_3,A_3} \mathrm{COS}(A)\mathrm{SIN}(A) + D_{F_1,A_1} \mathrm{COS}(A)\mathrm{SIN}(A)$$

$$+ D_{F_1,A_3} \mathrm{COS}^2(A))\mathrm{DEL}(A_3) - (D_{F_1,U_2} \mathrm{COS}(A) - D_{F_3,U_2} \mathrm{SIN}(A))\mathrm{DEL}(U_2)$$

$$- (D_{F_1,P_2} \mathrm{COS}(A) - D_{F_3,P_2} \mathrm{SIN}(A))\mathrm{DEL}(P_2) - (D_{F_1,A_2} \mathrm{COS}(A)$$

$$- D_{F_3,A_2} \mathrm{SIN}(A))\mathrm{DEL}(A_2) - (D_{F_3,U_3} \mathrm{SIN}^2(A) - D_{F_3,U_1} \mathrm{COS}(A)\mathrm{SIN}(A)$$

$$- D_{F_1,U_3} \mathrm{COS}(A)\mathrm{SIN}(A) + D_{F_1,U_1} \mathrm{COS}^2(A))\mathrm{DEL}(U_1) - (D_{F_3,P_3} \mathrm{SIN}^2(A)$$

$$- D_{F_3,P_1} \cos(A)\sin(A) - D_{F_1,P_3} \cos(A)\sin(A)$$

$$+ D_{F_1,P_1} \cos^2(A))\mathrm{DEL}(P_1) - (D_{F_3,A_3} \sin^2(A) - D_{F_3,A_1} \cos(A)\sin(A)$$

$$- D_{F_1,A_3} \cos(A)\sin(A) + D_{F_1,A_1} \cos^2(A))\mathrm{DEL}(A_1) + \sin(R_2)GM$$

$$+ (\frac{d}{dT} U_1 + P_2U_3 - U_2P_3)M + S_{F_3} \sin(A) - S_{F_1} \cos(A)$$

$$FT_2 = -D_{F_2,C_K} \mathrm{DEL}(C_K) - (D_{F_2,U_1} \sin(A) + D_{F_2,U_3} \cos(A))\mathrm{DEL}(U_3)$$

$$- (D_{F_2,P_1} \sin(A) + D_{F_2,P_3} \cos(A))\mathrm{DEL}(P_3) - (D_{F_2,A_1} \sin(A)$$

$$+ D_{F_2,A_3} \cos(A))\mathrm{DEL}(A_3) - D_{F_2U_2} \mathrm{DEL}(U_2) - D_{F_2,P_2} \mathrm{DEL}(P_2)$$

$$- D_{F_2,A_2} \mathrm{DEL}(A_2) - (D_{F_2,U_1} \cos(A) - D_{F_2,U_3} \sin(A))\mathrm{DEL}(U_1)$$

$$- (D_{F_2,P_1} \cos(A) - D_{F_2,P} \sin(A))\ \mathrm{DEL}(P_1) - (D_{F_2,A_1} \cos(A)$$

$$- D_{F_2,A_3} \sin(A))\mathrm{DEL}(A_1) - \sin(R_1)\cos(R_2)GM$$

$$+ (\frac{d}{dT} U_2 - P_1U_3 + U_1P_3)M - S_{F_2}$$

$$FT_3 = -(D_{F_1,C_K} \sin(A) + D_{F_3,C_K} \cos(A))\mathrm{DEL}(C_K)$$

$$- (D_{F_1,U_1} \sin^2(A) + D_{F_3,U_1} \cos(A)\sin(A) + D_{F_1,U_3}\cos(A)\sin(A)$$

$$+ D_{F_3,U_3} \cos^2(A))\mathrm{DEL}(U_3) - (D_{F_1,P_1} \sin^2(A) + D_{F_3,P_1} \cos(A)\sin(A)$$

$$+ D_{F_1,P_3} \cos(A)\sin(A) + D_{F_3,P_3} \cos^2(A))\mathrm{DEL}(P_3) - (D_{F_1,A_1} \sin^2(A)$$

$$+ D_{F_3,A_1} \cos(A)\sin(A) + D_{F_1,A_3} \cos(A)\sin(A)$$

$$+ D_{F_3,A_3} \cos^2(A))\mathrm{DEL}(A_3) - (D_{F_1,U_2} \sin(A) + D_{F_3,U_2} \cos(A))\mathrm{DEL}(U_2)$$

$$- (D_{F_1,P_2} \mathrm{SIN}(A) + D_{F_3,P_2} \mathrm{COS}(A))\mathrm{DEL}(P_2) - (D_{F_1,A_2} \mathrm{SIN}(A)$$

$$+ D_{F_3,A_2} \mathrm{COS}(A))\mathrm{DEL}(A_2) - (-D_{F_1,U_3} \mathrm{SIN}^2(A) - D_{F_3,U_3} \mathrm{COS}(A)\mathrm{SIN}(A)$$

$$+ D_{F_1,U_1} \mathrm{COS}(A)\mathrm{SIN}(A) + D_{F_3,U_1} \mathrm{COS}^2(A))\mathrm{DEL}(U_1)$$

$$- (-D_{F_1,P_3} \mathrm{SIN}^2(A) - D_{F_3,P_3} \mathrm{COS}(A)\mathrm{SIN}(A) + D_{F_1,P_1} \mathrm{COS}(A)\mathrm{SIN}(A)$$

$$+ D_{F_3,P_1} \mathrm{COS}^2(A))\mathrm{DEL}(P_1) - (-D_{F_1,A_3} \mathrm{SIN}^2(A)$$

$$- D_{F_3,A_3} \mathrm{COS}(A)\mathrm{SIN}(A) + D_{F_1,A_1} \mathrm{COS}(A)\mathrm{SIN}(A)$$

$$+ D_{F_3,A_1} \mathrm{COS}^2(A))\mathrm{DEL}(A_1) + M(-U_1 P_2 + \frac{d}{dT} U_3 + P_1 U_2)$$

$$- \mathrm{COS}(R_1)\mathrm{COS}(R_2)GM - S_{F_1} \mathrm{SIN}(A) - S_{F_3} \mathrm{COS}(A)$$

In addition to the zero sideslip condition, the investigator might wish to determine vehicle response when the angle of attack $A$ is limited to small values. For this condition the program would request MACSYMA to make the substitution SIN(A) = A. Moreover, if the angle of attack were sufficiently small, the program would request MACSYMA to make the additional substitution COS(A) = 1.

In this case, the required substitutions and display commands give rise to the following modified equations:

```
(C50) FOR I:1 THRU 3 DO FT[I]:SUBST([SIN(A)=A,COS(A)=1],FT[I])$

(C51) FOR I:1 THRU 3 DO DISPLAY(FT[I])$
```

$$FT_1 = -(D_{F_1,C_K} - D_{F_3,C_K} A)\mathrm{DEL}(C_K) - (-D_{F_3,U_1} A^2 - D_{F_3,U_3} A$$

$$+ D_{F_1,U_1} A + D_{F_1,U_3})\mathrm{DEL}(U_3) - (-D_{F_3,P_1} A^2 - D_{F_3,P_3} A + D_{F_1,P_1} A$$

$$+ D_{F_1,P_3})\mathrm{DEL}(P_3) - (-D_{F_3,A_1} A^2 - D_{F_3,A_3} A + D_{F_1,A_1} A$$

$$+ D_{F_1,A_3})\mathrm{DEL}(A_3) - (D_{F_1,U_2} - D_{F_3,U_2} A)\mathrm{DEL}(U_2) - (D_{F_1,P_2}$$

$$- D_{F_3,P_2} A)DEL(P_2) - (D_{F_1,A_2} - D_{F_3,A_2} A)DEL(A_2) - (D_{F_3,U_3} A^2$$

$$- D_{F_3,U_1} A - D_{F_1,U_3} A + D_{F_1,U_1})DEL(U_1) - (D_{F_3,P_3} A^2 - D_{F_3,P_1} A$$

$$- D_{F_1,P_3} A + D_{F_1,P_1})DEL(P_1) - (D_{F_3,A_3} A^2 - D_{F_3,A_1} A - D_{F_1,A_3} A$$

$$+ D_{F_1,A_1})DEL(A_1) + SIN(R_2)GM + (\frac{d}{dT} U_1 + P_2U_3 - U_2P_3)M + S_{F_3} A - S_{F_1}$$

$$FT_2 = -D_{F_2,C_K} DEL(C_K) - (D_{F_2,U_1} A + D_{F_2,U_3})DEL(U_3) - (D_{F_2,P_1} A$$

$$+ D_{F_2,P_3})DEL(P_3) - (D_{F_2,A_1} A + D_{F_2,A_3})DEL(A_3) - D_{F_2,U_2} DEL(U_2)$$

$$- D_{F_2,P_2} DEL(P_2) - D_{F_2,A_2} DEL(A_2) - (D_{F_2,U_1} - D_{F_2,U_3} A)DEL(U_1)$$

$$- (D_{F_2,P_1} - D_{F_2,P_3} A)DEL(P_1) - (D_{F_2,A_1} - D_{F_2,A_3} A)DEL(A_1)$$

$$- SIN(R_1)COS(R_2)GM + (\frac{d}{dT} U_2 - P_1U_3 + U_1P_3)M - S_{F_2}$$

$$FT_3 = -(D_{F_1,C_K} A + D_{F_3,C_K})DEL(C_K) - (D_{F_1,U_1} A^2 + D_{F_3,U_1} A$$

$$+ D_{F_1,U_3} A + D_{F_3,U_3})DEL(U_3) - (D_{F_1,P_1} A^2 + D_{F_3,P_1} A + D_{F_1,P_3} A$$

$$+ D_{F_3,P_3})DEL(P_3) - (D_{F_1,A_1} A^2 + D_{F_3,A_1} A + D_{F_1,A_3} A$$

$$+ D_{F_3,A_3})DEL(A_3) - (D_{F_1,U_2} A + D_{F_3,U_2})DEL(U_2) - (D_{F_1,P_2} A$$

$$+ D_{F_3,P_2})DEL(P_2) - (D_{F_1,A_2} A + D_{F_3,A_2})DEL(A_2) - (-D_{F_1,U_3} A^2$$

$$- D_{F_3,U_3} A + D_{F_1,U_1} A + D_{F_3,U_1})DEL(U_1) - (-D_{F_1,P_3} A^2 - D_{F_3,P_3} A$$

$$+ D_{F_1,P_1} A + D_{F_3,P_1})DEL(P_1) - (-D_{F_1,A_3} A^2 - D_{F_3,A_3} A + D_{F_1,A_1} A$$

$$+ D_{F_3,A_1})DEL(A_1) + M(-U_1P_2 + \frac{d}{dT} U_3 + P_1U_2) - COS(R_1)COS(R_2)GM$$

$$- S_{F_1} A - S_{F_3}$$

Examination of these equations reveals the existence of terms such as $A^2$. If it is assumed that second-order terms in $A$ are negligible, a program statement instructing MACSYMA to make the substitution $A^2 = 0$ would simplify the equations as follows:

```
(C52) FOR I:1 THRU 3 DO FT[I]:SUBST([A**2=0],FT[I])$

(C53) FOR I:1 THRU 3 DO DISPLAY(FT[I])$
```

$$FT_1 = -(D_{F_1,C_K} - D_{F_3,C_K} A)DEL(C_K) - (-D_{F_3,U_3} A + D_{F_1,U_1} A$$

$$+ D_{F_1,U_3})DEL(U_3) - (-D_{F_3,P_3} A + D_{F_1,P_1} A + D_{F_1,P_3})DEL(P_3)$$

$$- (-D_{F_3,A_3} A + D_{F_1,A_1} A + D_{F_1,A_3})DEL(A_3) - (D_{F_1,U_2}$$

$$- D_{F_3,U_2} A)DEL(U_2) - (D_{F_1,P_2} - D_{F_3,P_2} A)DEL(P_2) - (D_{F_1,A_2}$$

$$- D_{F_3,A_2} A)DEL(A_2) - (-D_{F_3,U_1} A - D_{F_1,U_3} A + D_{F_1,U_1})DEL(U_1)$$

$$- (-D_{F_3,P_1} A - D_{F_1,P_3} A + D_{F_1,P_1})DEL(P_1) - (-D_{F_3,A_1} A - D_{F_1,A_3} A$$

$$+ D_{F_1,A_1})DEL(A_1) + SIN(R_2)GM + (\frac{d}{dT} U_1 + P_2U_3 - U_2P_3)M + S_{F_3} A - S_{F_1}$$

$$FT_2 = -D_{F_2,C_K} DEL(C_K) - (D_{F_2,U_1} A + D_{F_2,U_3})DEL(U_3) - (D_{F_2,P_1} A$$

$$+ D_{F_2,P_3})DEL(P_3) - (D_{F_2,A_1} A + D_{F_2,A_3})DEL(A_3) - D_{F_2,U_2} DEL(U_2)$$

$$- D_{F_2,P_2} DEL(P_2) - D_{F_2,A_2} DEL(A_2) - (D_{F_2,U_1} - D_{F_2,U_3} A)DEL(U_1)$$

$$- (D_{F_2,P_1} - D_{F_2,P_3} A)DEL(P_1) - (D_{F_2,A_1} - D_{F_2,A_3} A)DEL(A_1)$$

$$- SIN(R_1)COS(R_2)GM + (\frac{d}{dT} U_2 - P_1U_3 + U_1P_3)M - S_{F_2}$$

$$FT_3 = -(D_{F_1,C_K} A + D_{F_3,C_K}) DEL(C_K) - (D_{F_3,U_1} A + D_{F_1,U_3} A + D_{F_3,U_3})$$

$$DEL(U_3) - (D_{F_3,P_1} A + D_{F_1,P_3} A + D_{F_3,P_3}) DEL(P_3) - (D_{F_3,A_1} A$$

$$+ D_{F_1,A_3} A + D_{F_3,A_3}) DEL(A_3) - (D_{F_1,U_2} A + D_{F_3,U_2}) DEL(U_2)$$

$$- (D_{F_1,P_2} A + D_{F_3,P_2}) DEL(P_2) - (D_{F_1,A_2} A + D_{F_3,A_2}) DEL(A_2)$$

$$- (-D_{F_3,U_3} A + D_{F_1,U_1} A + D_{F_3,U_1}) DEL(U_1) - (-D_{F_3,P_3} A + D_{F_1,P_1} A$$

$$+ D_{F_3,P_1}) DEL(P_1) - (-D_{F_3,A_3} A + D_{F_1,A_1} A + D_{F_3,A_1}) DEL(A_1)$$

$$+ M(-U_1 P_2 + \frac{d}{dT} U_3 + P_1 U_2) - COS(R_1) COS(R_2) GM - S_{F_1} A - S_{F_3}$$

Additional simplifications are possible if it is assumed that angular velocity perturbations are negligible. This assumption can be implemented by again using the substitution command, which yields the following greatly simplified equations:

```
(C54) FOR I:1 THRU 3 DO FOR J:1 THRU 3 DO
 FT[I]:SUBST([DEL(P[J])=0],FT[I])$

(C56) FOR I:1 THRU 3 DO DISPLAY(FT[I])$
```

$$FT_1 = -(D_{F_1,C_K} - D_{F_3,C_K} A) DEL(C_K) - (-D_{F_3,U_3} A + D_{F_1,U_1} A$$

$$+ D_{F_1,U_3}) DEL(U_3) - (-D_{F_3,A_3} A + D_{F_1,A_1} A + D_{F_1,A_3}) DEL(A_3)$$

$$- (D_{F_1,U_2} - D_{F_3,U_2} A) DEL(U_2) - (D_{F_1,A_2} - D_{F_3,A_2} A) DEL(A_2)$$

$$- (-D_{F_3,U_1} A - D_{F_1,U_3} A + D_{F_1,U_1}) DEL(U_1) - (-D_{F_3,A_1} A - D_{F_1,A_3} A$$

$$+ D_{F_1,A_1}) DEL(A_1) + SIN(R_2) GM + (\frac{d}{dT} U_1 + P_2 U_3 - U_2 P_3) M + S_{F_3} A - S_{F_1}$$

$$FT_2 = -D_{F_2,C_K} DEL(C_K) - (D_{F_2,U_1} A + D_{F_2,U_3}) DEL(U_3) - (D_{F_2,A_1} A$$

$$+ D_{F_2,A_3}) DEL(A_3) - D_{F_2,U_2} DEL(U_2) - D_{F_2,A_2} DEL(A_2)$$

$$- (D_{F_2,U_1} - D_{F_2,U_3}\, A)DEL(U_1) - (D_{F_2,A_1} - D_{F_2,A_3}\, A)DEL(A_1)$$

$$- SIN(R_1)COS(R_2)GM + (\frac{d}{dT} U_2 - P_1 U_3 + U_1 P_3)M - S_{F_2}$$

$$FT_3 = -(D_{F_1,C_K}\, A + D_{F_3,C_K})DEL(C_K) - (D_{F_3,U_1}\, A + D_{F_1,U_3}\, A + D_{F_3,U_3})$$

$$DEL(U_3) - (D_{F_3,A_1}\, A + D_{F_1,A_3}\, A + D_{F_3,A_3})DEL(A_3) - (D_{F_1,U_2}\, A$$

$$+ D_{F_3,U_2})DEL(U_2) - (D_{F_1,A_2}\, A + D_{F_3,A_2})DEL(A_2) - (-D_{F_3,U_3}\, A$$

$$+ D_{F_1,U_1}\, A + D_{F_3,U_1})DEL(U_1) - (-D_{F_3,A_3}\, A + D_{F_1,A_1}\, A$$

$$+ D_{F_3,A_1})DEL(A_1) + M(-U_1 P_2 + \frac{d}{dT} U_3 + P_1 U_2)$$

$$- COS(R_1)COS(R_2)GM - S_{F_1}\, A - S_{F_3}$$

Finally, it may be of interest to consider the effect of omitting the linear acceleration terms. By comparing the response of the system with and without acceleration perturbations, the influence of these perturbations can be determined. Again, a simple substitution command is all that is required to implement the assumption that $DEL(A_i)=0$. Execution of this command yields the modified equations as follows:

```
(C57) FOR I:1 THRU 3 DO FOR J:1 THRU 3 DO
 FT[I]:SUBST([DEL(A[J])=0],FT[I])$

(C58) FOR I:1 THRU 3 DO DISPLAY(FT[I])$
```

$$FT_1 = -(D_{F_1,C_K} - D_{F_3,C_K}\, A)DEL(C_K) - (-D_{F_3,U_3}\, A + D_{F_1,U_1}\, A$$

$$+ D_{F_1,U_3})DEL(U_3) - (D_{F_1,U_2} - D_{F_3,U_2}\, A)DEL(U_2) - (-D_{F_3,U_1}\, A$$

$$- D_{F_1,U_3}\, A + D_{F_1,U_1})DEL(U_1) + SIN(R_2)GM + (\frac{d}{dT} U_1 + P_2 U_3 - U_2 P_3)M$$

$$+ S_{F_3}\, A - S_{F_1}$$

$$FT_2 = -D_{F_2,C_K}\ DEL(C_K) - (D_{F_2,U_1}\ A + D_{F_2,U_3})DEL(U_3) - D_{F_2,U_2}\ DEL(U_2)$$

$$- (D_{F_2,U_1} - D_{F_2,U_3}\ A)DEL(U_1) - SIN(R_1)COS(R_2)GM$$

$$+ (\frac{d}{dT}\ U_2 - P_1U_3 + U_1P_3)M - S_{F_2}$$

$$FT_3 = -(D_{F_1,C_K}\ A + D_{F_3,C_K})DEL(C_K) - (D_{F_3,U_1}\ A + D_{F_1,U_3}\ A + D_{F_3,U_3})$$

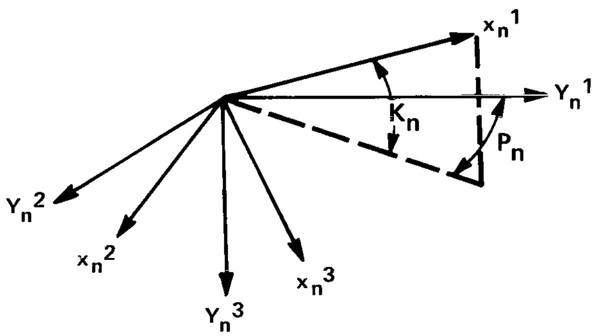
$$DEL(U_3) - (D_{F_1,U_2}\ A + D_{F_3,U_2})DEL(U_2) - (-D_{F_3,U_3}\ A + D_{F_1,U_1}\ A$$

$$+ D_{F_3,U_1})DEL(U_1) + M(-U_1P_2 + \frac{d}{dT}\ U_3 + P_1U_2)$$

$$- COS(R_1)COS(R_2)GM - S_{F_1}\ A - S_{F_3}$$

## Thrust Forces

It should be noted that the thrust forces $FT_i$ appearing on the left-hand side of these equations are the resultant of a number of thrust generating systems, each contributing a thrust vector $T_n$. Each thrust vector is referred to a thrust axes system $X_n{}^i$ with origin at the point of application of the thrust vector. The axes are chosen such that each thrust vector coincides with the $X_n{}^1$ axis of the system. Moreover, each thrust vector is then transformed to a coordinate system $Y_n{}^i$ which has the same origin as the thrust axes, but is parallel to the body axes system. Finally, the components of thrust in the $Y_n{}^i$ system of axes are transformed to the body axes system, which has its origin at the center of gravity of the aircraft. Each thrust axis $X_n{}^1$ is related to the $Y_ni$ system by the following transformation equations (see sketch):




$$\left.\begin{aligned} Y_n{}^1 &= X_n{}^1 \cos(K_n)\cos(P_n) \\ Y_n{}^2 &= X_n{}^1 \cos(K_n)\sin(P_n) \\ Y_n{}^3 &= -X_n{}^1 \sin(K_n) \end{aligned}\right\} \qquad (13)$$

Hence, the components of the thrust vector $T_n$ in the $Y_n{}^i$ system of coordinates are

$$\frac{\partial Y_n{}^1}{\partial X_n{}^1}\,T_n\ ;\quad \frac{\partial Y_n{}^2}{\partial X_n{}^1}\,T_n\ ;\quad \frac{\partial Y_n{}^3}{\partial X_n{}^1}\,T_n \qquad (14)$$

These are also the components of thrust in the $y^i$ system of coordinates, which has its origin at the center of gravity of the aircraft. The thrust components due to all thrust generating systems are obtained by summing the right-hand side of the following equation

$$T^i = \frac{\partial Y_n{}^i}{\partial X_n{}^1} T_n \tag{15}$$

The expanded form of equation (15), when summed over $n$ will yield the resultant thrust components. When the number of thrust generating systems is known, the components $T^i$ can be formulated and displayed by using equation (15), and executing the following two commands, which yield the components contributed by the $n$th thrust generating system. These are

```
(C1) Y[1,N]:X[1,N]*COS(K[N])*COS(P[N])$

(C2) Y[2,N]:X[1,N]*COS(K[N])*SIN(P[N])$

(C3) Y[3,N]:-X[1,N]*SIN(K[N])$

(C4) FOR I THRU 3 DO T[I]:DIFF(Y[I,N],X[1,N],1)*T[N]$

(C5) FOR I:1 THRU 3 DO DISPLAY (T[I])$
```

$$\left.\begin{aligned} T_1 &= T_N \cos(K_N) \cos(P_N) \\ T_2 &= T_N \cos(K_N) \sin(P_N) \\ T_3 &= -T_N \sin(K_N) \end{aligned}\right\} \tag{16}$$

Determination of the Geographical Location of Aircraft

In order to determine the geographical location of an aircraft relative to some initial location, it is necessary to transform the components of the aircraft's velocity vector from aircraft body axes to a system of earth-fixed axes. The transformed components can then be integrated to find the location of the aircraft as a function of time. The product of the three rotation matrices (D33), (D32), and (D31), which were used to transform the gravity vector from an earth-fixed axes system to aircraft body axes, may be transposed and used to transform the aircraft velocity components to an earth-fixed system. If the column vector (D39) of aircraft velocity components is premultiplied by the transposed matrix, the velocity components relative to the earth-fixed system are obtained as follows:

```
(C56) TRANSPOSE((D33).(D32).(D31)).(D39);
```

```
(D56) MATRIX([U3(SIN(R1)SIN(R3) + COS(R1)SIN(R2)COS(R3))
 + U2(SIN(R1)SIN(R2)COS(R3) - COS(R1)SIN(R3)) + U1COS(R2)COS(R3)],
[U2(SIN(R1)SIN(R2)SIN(R3) + COS(R1)COS(R3))
 + U3(COS(R1)SIN(R2)SIN(R3) - SIN(R1)COS(R3)) + U1COS(R2)SIN(R3)],
[-U1SIN(R2) + U2SIN(R1)COS(R2) + U3COS(R1)COS(R2)])
```

If the components $\dot{X}_i$ relative to the Earth-fixed system be denoted by $DX_i$, execution of the following programming steps will ensure that the required velocity components are displayed in conventional form.

```
(C57) FOR I:1 THRU 3 DO ROW[I]:FIRST(ROW((D56),I))$
(C58) FOR I:1 THRU 3 DO (DX[I]:ROW[I][1],DISPLAY(DX[I]));
DX1 = U3(SIN(R1)SIN(R3) + COS(R1)SIN(R2)COS(R3))
 + U2(SIN(R1)SIN(P2)COS(R3) - COS(R1)SIN(R3)) + U1COS(R2)COS(R3)
DX2 = U2(SIN(R1)SIN(R2)SIN(R3) + COS(R1)COS(R3))
 + U3(COS(R1)SIN(R2)SIN(R3) - SIN(R1)COS(R3)) + U1COS(R2)SIN(R3)
DX3 = -U1SIN(R2) + U2SIN(R1)COS(R2) + U3COS(R1)COS(R2)
```

Integration of these velocity components will yield the required coordinates of the aircraft relative to a set of earth-fixed reference axes. These are

$$X_E^{\,i} = X^i_{EO} + \int DX[I]\,dt$$

where $X^i_{EO}$ are the initial values of the coordinates in the earth-fixed reference frame.

## MOMENTS

### Transformation of Static Moments

The static aerodynamic moments obey the same transformation law as the static aerodynamic forces; that is, if $S_{M_n}$ denotes a static moment in the $X$ frame of reference, and $SM_i$ denotes the corresponding transformed moment in the $Y$ reference frame, then

$$SM_i = \frac{\partial Y^i}{\partial X^n} S_{M_n} \tag{17}$$

where $Y = Y(X)$ is obtained from the displayed output (D6) and reentered here to facilitate the formulation of the moment equations. Given the transformation equations (D6), the transformed aerodynamic static moments are obtained by expanding equation (17). The three programming steps used to transform the static forces may again be employed to transform the static moments. The simple program and the displayed results are

```
(C1) Y[1]:X[1]*COS(A)*COS(B)-X[2]*COS(A)*SIN(B)-X[3]*SIN(A)$

(C2) Y[2]:X[1]*SIN(B)+X[2]*COS(B)$

(C3) Y[3]:X[1]*SIN(A)*COS(B)-X[2]*SIN(A)*SIN(B)+X[3]*COS(A)$

(C4) SM[I]:=0$

(C5) FOR I:1 THRU 3 DO FOR N:1 THRU 3 DO
 SM[I]:SM[I]+DIFF(Y[I],X[N])*S[M[N]]$

(C6) FOR I:1 THRU 3 DO DISPLAY (SM[I])$
```

$$SM_1 = -S_{M_2} \, COS(A)SIN(B) + S_{M_1} \, COS(A)COS(B) - S_{M_3} \, SIN(A)$$

$$SM_2 = S_{M_1} \, SIN(B) + S_{M_2} \, COS(B)$$

$$SM_3 = -S_{M_2} \, SIN(A)SIN(B) + S_{M_1} \, SIN(A)COS(B) + S_{M_3} \, COS(A)$$

Transformation of Control Moment Derivatives

The control moment derivatives obey the same transformation law as the static moments; that is, if $D_{M_n,C_K}$ denotes the $n$th control moment derivative with respect to the $K$th control surface as measured in the $X$ reference frame, and $TD_{I,C}$ denotes the corresponding transformed derivative in the $Y$ frame, then

$$TD_{I,C} = \frac{\partial Y^i}{\partial X^n} D_{M_n,C_k} \tag{18}$$

where $Y = Y(X)$ is again obtained from the displayed output (D6).

As in the preceding section, the transformed control derivatives are obtained by expanding the transformation law (18) given the transformation equations (D6). The transformed derivatives are obtained by executing the following simple program, which has exactly the same form as the program used to transform the static moments. These are

```
(C7) TD[I,C]:=0$

(C8) FOR I:1 THRU 3 DO FOR N:1 THRU 3 DO
 TD[I,C]:TD[I,C]+DIFF(Y[I],X[N])*D[M[N],C[K]]$

(C9) FOR I:1 THRU 3 DO DISPLAY(TD[I,C])$
```

$$TD_{1,C} = -D_{M_2,C_K} \, COS(A)SIN(B) + D_{M_1,C_K} \, COS(A)COS(B) - D_{M_3,C_K} \, SIN(A)$$

$$TD_{2,C} = D_{M_1,C_K} \, SIN(B) + D_{M_2,C_K} \, COS(B)$$

$$TD_{3,C} = -D_{M_2,C_K} \, SIN(A)SIN(B) + D_{M_1,C_K} \, SIN(A)COS(B) + D_{M_3,C_K} \, COS(A)$$

The corresponding control moments are obtained by multiplying the control derivatives by the appropriate control increments $DEL(C_K)$. The following two programming steps are sufficient to formulate the required moments. These are denoted by $CM_i$ in the displayed output.

```
(C10) FOR I:1 THRU 3 DO CM[I]:TD[I,C]*DEL(C[K])$

(C11) FOR I:1 THRU 3 DO DISPLAY(CM[I])$
```

$$CM_1 = (-D_{M_2,C_K} \, COS(A)SIN(B) + D_{M_1,C_K} \, COS(A)COS(B) - D_{M_3,C_K} \, SIN(A))DEL(C_K)$$

$$CM_2 = (D_{M_1,C_K} \, SIN(B) + D_{M_2,C_K} \, COS(B))DEL(C_K)$$

$$CM_3 = (-D_{M_2,C_K} \, SIN(A)SIN(B) + D_{M_1,C_K} \, SIN(A)COS(B) + D_{M_3,C_K} \, COS(A))DEL(C_K)$$

### Moments Produced by Linear Velocity Perturbations

The next step in the formulation involves the determination of the aerodynamic moments produced when an aircraft is subjected to linear velocity perturbations $DEL(U_J)$. Before these moments can be determined, the aerodynamic stability derivatives with respect to linear velocity components must be transformed from wind or wind-tunnel stability axes to body axes. For a detailed discussion of the transformation of these derivatives, the reader is referred to equations (4) through (10). The program used for the transformation of force derivatives can be used in this case also. In this application, the aerodynamic stability derivative of the $i$th moment with respect to the $j$th velocity component will be denoted by $D_{M_i,U_j}$. The corresponding transformed derivatives are denoted by $TD_{M_i,U_j}$. When the program is rewritten to accommodate the notational changes required for this application, it assumes the following form:

```
(C12) TDU[I,J]:=0$

(C13) FOR I:1 THRU 3 DO FOR J:1 THRU 3 DO
 FOR R:1 THRU 3 DO FOR N:1 THRU 3 DO
 TDU[I,J]:TDU[I,J]+DIFF(Y[I],X[R])*DIFF(Y[J],X[N])*D[M[R],U[N]]$
```

It only remains to multiply the transformed derivatives by the appropriate velocity increments to obtain the required moments, which are denoted by $MDU_i$. The next three programming steps instruct MACSYMA to evaluate and display the moments produced by linear velocity perturbations. These are

```
(C14) MDU[I]:=0$

(C15) FOR I:1 THRU 3 DO FOR J:1 THRU 3 DO
 MDU[I]:MDU[I]+TDU[I,J]*DEL(U[J])$

(C16) FOR I:1 THRU 3 DO DISPLAY(MDU[I])$
```

$$
\begin{aligned}
\mathrm{MDU}_1 = (&D_{M_2,U_2}\ \mathrm{COS(A)SIN(A)SIN^2(B)} \\
&- D_{M_2,U_1}\ \mathrm{COS(A)SIN(A)COS(B)SIN(B)} \\
&- D_{M_1,U_2}\ \mathrm{COS(A)SIN(A)COS(B)SIN(B)} + D_{M_3,U_2}\ \mathrm{SIN^2(A)SIN(B)} \\
&- D_{M_2,U_3}\ \mathrm{COS^2(A)SIN(B)} + D_{M_1,U_1}\ \mathrm{COS(A)SIN(A)COS^2(B)} \\
&- D_{M_3,U_1}\ \mathrm{SIN^2(A)COS(B)} + D_{M_1,U_3}\ \mathrm{COS^2(A)COS(B)} \\
&- D_{M_3,U_3}\ \mathrm{COS(A)SIN(A))DEL(U_3)} + (-D_{M_2,U_1}\ \mathrm{COS(A)SIN^2(B)} \\
&- D_{M_2,U_2}\ \mathrm{COS(A)COS(B)SIN(B)} + D_{M_1,U_1}\ \mathrm{COS(A)COS(B)SIN(B)} \\
&- D_{M_3,U_1}\ \mathrm{SIN(A)SIN(B)} + D_{M_1,U_2}\ \mathrm{COS(A)COS^2(B)} \\
&- D_{M_3,U_2}\ \mathrm{SIN(A)COS(B))DEL(U_2)} + (D_{M_2,U_2}\ \mathrm{COS^2(A)SIN^2(B)}
\end{aligned}
$$

$$- D_{M_2,U_1} \cos^2(A)\cos(B)\sin(B) - D_{M_1,U_2} \cos^2(A)\cos(B)\sin(B)$$

$$+ D_{M_3,U_2} \cos(A)\sin(A)\sin(B) + D_{M_2,U_3} \cos(A)\sin(A)\sin(B)$$

$$+ D_{M_1,U_1} \cos^2(A)\cos^2(B) - D_{M_3,U_1} \cos(A)\sin(A)\cos(B)$$

$$- D_{M_1,U_3} \cos(A)\sin(A)\cos(B) + D_{M_3,U_3} \sin^2(A))\mathrm{DEL}(U_1)$$

$$\mathrm{MDU}_2 = (-D_{M_1,U_2} \sin(A)\sin^2(B) - D_{M_2,U_2} \sin(A)\cos(B)\sin(B)$$

$$+ D_{M_1,U_1} \sin(A)\cos(B)\sin(B) + D_{M_1,U_3} \cos(A)\sin(B)$$

$$+ D_{M_2,U_1} \sin(A)\cos^2(B) + D_{M_2,U_3} \cos(A)\cos(B))\mathrm{DEL}(U_3)$$

$$+ (D_{M_1,U_1} \sin^2(B) + D_{M_2,U_1} \cos(B)\sin(B) + D_{M_1,U_2} \cos(B)\sin(B)$$

$$+ D_{M_2,U_2} \cos^2(B))\mathrm{DEL}(U_2) + (-D_{M_1,U_2} \cos(A)\sin^2(B)$$

$$- D_{M_2,U_2} \cos(A)\cos(B)\sin(B) + D_{M_1,U_1} \cos(A)\cos(B)\sin(B)$$

$$- D_{M_1,U_3} \sin(A)\sin(B) + D_{M_2,U_1} \cos(A)\cos^2(B)$$

$$- D_{M_2,U_3} \sin(A)\cos(B))\mathrm{DEL}(U_1)$$

$$\mathrm{MDU}_3 = (D_{M_2,U_2} \sin^2(A)\sin^2(B) - D_{M_2,U_1} \sin^2(A)\cos(B)\sin(B)$$

$$- D_{M_1,U_2} \sin^2(A)\cos(B)\sin(B) - D_{M_3,U_2} \cos(A)\sin(A)\sin(B)$$

$$- D_{M_2,U_3} \cos(A)\sin(A)\sin(B) + D_{M_1,U_1} \sin^2(A)\cos^2(B)$$

$$+ D_{M_3,U_1} \cos(A)\sin(A)\cos(B) + D_{M_1,U_3} \cos(A)\sin(A)\cos(B)$$

$$+ D_{M_3,U_3} \cos^2(A))\mathrm{DEL}(U_3) + (-D_{M_2,U_1} \sin(A)\sin^2(B)$$

$- D_{M_2,U_2}$ SIN(A)COS(B)SIN(B) $+ D_{M_1,U_1}$ SIN(A)COS(B)SIN(B)

$+ D_{M_3,U_1}$ COS(A)SIN(B) $+ D_{M_1,U_2}$ SIN(A)COS$^2$(B)

$+ D_{M_3,U_2}$ COS(A)COS(B))DEL($U_2$) + ($D_{M_2,U_2}$ COS(A)SIN(A)SIN$^2$(B)

$- D_{M_2,U_1}$ COS(A)SIN(A)COS(B)SIN(B)

$- D_{M_1,U_2}$ COS(A)SIN(A)COS(B)SIN(B) $+ D_{M_2,U_3}$ SIN$^2$(A)SIN(B)

$- D_{M_3,U_2}$ COS$^2$(A)SIN(B) $+ D_{M_1,U_1}$ COS(A)SIN(A)COS$^2$(B)

$- D_{M_1,U_3}$ SIN$^2$(A)COS(B) $+ D_{M_3,U_1}$ COS$^2$(A)COS(B)

$- D_{M_3,U_3}$ COS(A)SIN(A))DEL($U_1$)

## Moments Produced by Angular Velocity Perturbations

The program used in the preceding section can, with suitable notational changes, be used to formulate the moments produced by angular velocity perturbations. However, whereas in the preceding application the required moments were obtained by multiplying the transformed aerodynamic stability derivatives by linear velocity increments, in the present case the transformed derivatives must be multiplied by angular velocity increments. In view of these similarities, the following program and displayed moments will be presented without further comment, except to point out that the aerodynamic stability derivatives of the $i$th moment with respect to the $j$th angular velocity component are denoted by $D_{M_i,P_j}$. The corresponding transformed derivatives are denoted by $TD_{M_i,P_j}$, and the resulting moments by $MDP_i$.

```
(C17) TDP[I,J]:=0$

(C18) FOR I:1 THRU 3 DO FOR J:1 THRU 3 DO
 FOR R:1 THRU 3 DO FOR N:1 THRU 3 DO
 TDP[I,J]:TDP[I,J]+DIFF(Y[I],X[R])*DIFF(Y[J],X[N])*D[M[R],P[N]$

(C19) MDP[I]:=0$

(C21) FOR I:1 THRU 3 DO FOR J:1 THRU 3 DO
 MDP[I]:MDP[I]+TDP[I,J]*DEL(P[J])$

(C22) FOR I:1 THRU 3 DO DISPLAY(MDP[I])$
```

$$
\begin{aligned}
MDP_1 = (&D_{M_2,P_2}\ COS(A)SIN(A)SIN^2(B) \\
&- D_{M_2,P_1}\ COS(A)SIN(A)COS(B)SIN(B) \\
&- D_{M_1,P_2}\ COS(A)SIN(A)COS(B)SIN(B) + D_{M_3,P_2}\ SIN^2(A)SIN(B) \\
&- D_{M_2,P_3}\ COS^2(A)SIN(B) + D_{M_1,P_1}\ COS(A)SIN(A)COS^2(B) \\
&- D_{M_3,P_1}\ SIN^2(A)COS(B) + D_{M_1,P_3}\ COS^2(A)COS(B) \\
&- D_{M_3,P_3}\ COS(A)SIN(A))DEL(P_3) + (-D_{M_2,P_1}\ COS(A)SIN^2(B) \\
&- D_{M_2,P_2}\ COS(A)COS(B)SIN(B) + D_{M_1,P_1}\ COS(A)COS(B)SIN(B) \\
&- D_{M_3,P_1}\ SIN(A)SIN(B) + D_{M_1,P_2}\ COS(A)COS^2(B) \\
&- D_{M_3,P_2}\ SIN(A)COS(B))DEL(P_2) + (D_{M_2,P_2}\ COS^2(A)SIN^2(B) \\
&- D_{M_2,P_1}\ COS^2(A)COS(B)SIN(B) - D_{M_1,P_2}\ COS^2(A)COS(B)SIN(B) \\
&+ D_{M_3,P_2}\ COS(A)SIN(A)SIN(B) + D_{M_2,P_3}\ COS(A)SIN(A)SIN(B) \\
&+ D_{M_1,P_1}\ COS^2(A)COS^2(B) - D_{M_3,P_1}\ COS(A)SIN(A)COS(B) \\
&- D_{M_1,P_3}\ COS(A)SIN(A)COS(B) + D_{M_3,P_3}\ SIN^2(A))DEL(P_1)
\end{aligned}
$$

$$
\begin{aligned}
MDP_2 = (&-D_{M_1,P_2}\ SIN(A)SIN^2(B) - D_{M_2,P_2}\ SIN(A)COS(B)SIN(B) \\
&+ D_{M_1,P_1}\ SIN(A)COS(B)SIN(B) + D_{M_1,P_3}\ COS(A)SIN(B) \\
&+ D_{M_2,P_1}\ SIN(A)COS^2(B) + D_{M_2,P_3}\ COS(A)COS(B))DEL(P_3)
\end{aligned}
$$

$$+ (D_{M_1,P_1} \ SIN^2(B) + D_{M_2,P_1} \ COS(B)SIN(B) + D_{M_1,P_2} \ COS(B)SIN(B)$$

$$+ D_{M_2,P_2} \ COS^2(B))DEL(P_2) + (-D_{M_1,P_2} \ COS(A)SIN^2(B)$$

$$- D_{M_2,P_2} \ COS(A)COS(B)SIN(B) + D_{M_1,P_1} \ COS(A)COS(B)SIN(B)$$

$$- D_{M_1,P_3} \ SIN(A)SIN(B) + D_{M_2,P_1} \ COS(A)COS^2(B)$$

$$- D_{M_2,P_3} \ SIN(A)COS(B))DEL(P_1)$$

$$MDP_3 = (D_{M_2,P_2} \ SIN^2(A)SIN^2(B) - D_{M_2,P_1} \ SIN^2(A)COS(B)SIN(B)$$

$$- D_{M_1,P_2} \ SIN^2(A)COS(B)SIN(B) - D_{M_3,P_2} \ COS(A)SIN(A)SIN(B)$$

$$- D_{M_2,P_3} \ COS(A)SIN(A)SIN(B) + D_{M_1,P_1} \ SIN^2(A)COS^2(B)$$

$$+ D_{M_3,P_1} \ COS(A)SIN(A)COS(B) + D_{M_1,P_3} \ COS(A)SIN(A)COS(B)$$

$$+ D_{M_3,P_3} \ COS^2(A))DEL(P_3) + (-D_{M_2,P_1} \ SIN(A)SIN^2(B)$$

$$- D_{M_2,P_2} \ SIN(A)COS(B)SIN(B) + D_{M_1,P_1} \ SIN(A)COS(B)SIN(B)$$

$$+ D_{M_3,P_1} \ COS(A)SIN(B) + D_{M_1,P_2} \ SIN(A)COS^2(B)$$

$$+ D_{M_3,P_2} \ COS(A)COS(B))DEL(P_2) + (D_{M_2,P_2} \ COS(A)SIN(A)SIN^2(B)$$

$$- D_{M_2,P_1} \ COS(A)SIN(A)COS(B)SIN(B)$$

$$- D_{M_1,P_2} \ COS(A)SIN(A)COS(B)SIN(B) + D_{M_2,P_3} \ SIN^2(A)SIN(B)$$

$$- D_{M_3,P_2} \ COS^2(A)SIN(B) + D_{M_1,P_1} \ COS(A)SIN(A)COS^2(B)$$

$$- D_{M_1,P_3} \ SIN^2(A)COS(B) + D_{M_3,P_1} \ COS^2(A)COS(B)$$

$$- D_{M_3,P_3} \ COS(A)SIN(A))DEL(P_1)$$

The same procedure may be used to formulate the aerodynamic moments produced by linear and angular accelerations. These moments will not be included here, since the cases considered so far are sufficient to demonstrate the facility with which symbolic mathematical computation can be used to formulate and transform aerodynamic moments.

## Inertia Moments

The formulation of inertia moments involves the determination of the product of an angular velocity matrix, a matrix of inertia coefficients, and a column vector of angular velocity components. This product is the matrix equivalent of the familiar vector product $\bar{\omega} \times \bar{H}$, where $\bar{\omega}$ is the angular velocity vector and $\bar{H}$ is the angular momentum vector. By adding to the components of this vector, a vector which represents the rate of change of angular momentum relative to the moving body axes, the inertial moments relative to these axes are obtained. The rate of change of angular momentum relative to the moving body axes may be expressed as the product of the inertia matrix and a column vector of angular acceleration components. The required matrices may be entered and multiplied as follows: The first matrix to be entered is the inertia matrix, with elements $J_{i,j}$. It is entered by typing the statement ENTERMATRIX(3.3) and responding to the system's request for elements.

```
(C23) ENTERMATRIX(3,3);

ROW 1 COLUMN 1  J[1,1];

ROW 1 COLUMN 2  J[1,2];

ROW 1 COLUMN 3  J[1,3];

ROW 2 COLUMN 1  J[2,1];

ROW 2 COLUMN 2  J[2,2];

ROW 2 COLUMN 3  J[2,3];

ROW 3 COLUMN 1  J[3,1];

ROW 3 COLUMN 2  J[3,2];
```

```
ROW 3 COLUMN 3  J[3,3];

MATRIX-ENTERED
                              [ J     J     J    ]
                              [  1, 1  1, 2  1, 3 ]
(D23)                         [ J     J     J    ]
                              [  2, 1  2, 2  2, 3 ]
                              [ J     J     J    ]
                              [  3, 1  3, 2  3, 3 ]
```

A statement of the fact that the $i$th component of the angular velocity vector is a function of time requires the use of the DEPENDENCIES function. The use of this function permits the system to differentiate the components $P_i$ with respect to time, and to enter the resulting acceleration components in the form of a column vector as follows:

```
(C24) DEPENDENCIES(P(I,T))$

(C25) ENTERMATRIX(3,1);

ROW 1 COLUMN 1  DIFF(P[1],T);

ROW 2 COLUMN 1  DIFF(P[2],T);

ROW 3 COLUMN 1  DIFF(P[3],T);

MATRIX-ENTERED

                                   [ d       ]
                                   [ -- P    ]
                                   [ dT  1   ]
                                   [         ]
                                   [ d       ]
(D25)                              [ -- P    ]
                                   [ dT  2   ]
                                   [         ]
                                   [ d       ]
                                   [ -- P    ]
                                   [ dT  3   ]
```

The angular velocity matrix and a column vector of angular velocity components are entered next

```
(C26) ENTERMATRIX(3,3);

ROW 1 COLUMN 1  0;

ROW 1 COLUMN 2  -P[3];
```

```
ROW 1 COLUMN 3  P[2];

ROW 2 COLUMN 1  P[3];

ROW 2 COLUMN 2  0;

ROW 2 COLUMN 3  -P[1];

ROW 3 COLUMN 1  -P[2];

ROW 3 COLUMN 2  P[1];

ROW 3 COLUMN 3  0;

MATRIX-ENTERED
```

(D26)
$$\begin{bmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}$$

```
(C27) ENTERMATRIX(3,1);

ROW 1 COLUMN 1  P[1];

ROW 2 COLUMN 1  P[2];

ROW 3 COLUMN 1  P[3];

MATRIX-ENTERED
```

(D27)
$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

These four matrices are now combined to yield a column vector of inertia moments relative to aircraft body axes.

```
(C28) ((D23).(D25)+(D26).(D23).(D27));
```

(D28) MATRIX(
$$[J_{1,3} \left(\frac{d}{dT} P_3\right) + J_{1,2} \left(\frac{d}{dT} P_2\right) + J_{1,1} \left(\frac{d}{dT} P_1\right)$$

$$+ P_2(P_3J_{3,3} + P_2J_{3,2} + P_1J_{3,1}) - P_3(J_{2,3}P_3 + P_2J_{2,2} + P_1J_{2,1})],$$

$$[J_{2,3}\,(\frac{d}{dT}\,P_3) + J_{2,2}\,(\frac{d}{dT}\,P_2) + J_{2,1}\,(\frac{d}{dT}\,P_1) - P_1(P_3 J_{3,3} + P_2 J_{3,2}$$

$$+ P_1 J_{3,1}) + P_1 J_{3,1}) + P_3(J_{1,3} P_3 + J_{1,2} P_2 + P_1 J_{1,1})],\ [J_{3,3}\,(\frac{d}{dT}\,P_3)$$

$$+ J_{3,2}\,(\frac{d}{dT}\,P_2) + J_{3,1}\,(\frac{d}{dT}\,P_1) + P_1(J_{2,3} P_3 + P_2 J_{2,2} + P_1 J_{2,1})$$

$$- P_2(J_{1,3} P_3 + J_{1,2} P_2 + P_1 J_{1,1})]])$$

The next two programming steps enable the system to express these inertia moments in conventional functional form.

```
(C29) FOR I:1 THRU 3 DO ROW[I]:FIRST(ROW((D28),I))$

(C30) FOR I:1 THRU 3 DO (IM[I]:ROW[I][1],DISPLAY(IM[I]))$
```

$$IM_1 = J_{1,3}\,(\frac{d}{dT}\,P_3) + J_{1,2}\,(\frac{d}{dT}\,P_2) + J_{1,1}\,(\frac{d}{dT}\,P_1) + P_2(P_3 J_{3,3}$$

$$+ P_2 J_{3,2} + P_1 J_{3,1}) - P_3(J_{2,3} P_3 + P_2 J_{2,2} + P_1 J_{2,1})$$

$$IM_2 = J_{2,3}\,(\frac{d}{dT}\,P_3) + J_{2,2}\,(\frac{d}{dT}\,P_2) + J_{2,1}\,(\frac{d}{dT}\,P_1) - P_1(P_3 J_{3,3}$$

$$+ P_2 J_{3,2} + P_1 J_{3,1}) + P_3(J_{1,3} P_3 + J_{1,2} P_2 + P_1 J_{1,1})$$

$$IM_3 = J_{3,3}\,(\frac{d}{dT}\,P_3) + J_{3,2}\,(\frac{d}{dT}\,P_2) + J_{3,1}\,(\frac{d}{dT}\,P_1) + P_1(J_{2,3} P_3$$

$$+ P_2 J_{2,2} + P_1 J_{2,1}) - P_2(J_{1,3} P_3 + J_{1,2} P_2 + P_1 J_{1,1})$$

## Resultant Moments

It only remains to request MACSYMA to combine the aerodynamic and inertia moments which have been formulated in preceding sections and to display the results. The $i$th component of the resultant moment will be denoted by $TM_i$, where $TM_i$ is the $i$th component of the moment due to thrust. The two programming steps and the formulated equations follow.

```
(C31) FOR I:1 THRU 3 DO TM[I]:IM[I]-SM[I]-CM[I]-MDU[I]-MDP[I]$

(C32) FOR I:1 THRU 3 DO DISPLAY(TM[I])$
```

$$
\begin{aligned}
TM_1 = &-(-D_{M_2,C_K} \cos(A)\sin(B) + D_{M_1,C_K} \cos(A)\cos(B) \\
&- D_{M_3,C_K} \sin(A))DEL(C_K) - (D_{M_2,U_2} \cos(A)\sin(A)\sin^2(B) \\
&- D_{M_2,U_1} \cos(A)\sin(A)\cos(B)\sin(B) \\
&- D_{M_1,U_2} \cos(A)\sin(A)\cos(B)\sin(B) + D_{M_3,U_2} \sin^2(A)\sin(B) \\
&- D_{M_2,U_3} \cos^2(A)\sin(B) + D_{M_1,U_1} \cos(A)\sin(A)\cos^2(B) \\
&- D_{M_3,U_1} \sin^2(A)\cos(B) + D_{M_1,U_3} \cos^2(A)\cos(B) \\
&- D_{M_3,U_3} \cos(A)\sin(A))DEL(U_3) - (D_{M_2,P_2} \cos(A)\sin(A)\sin^2(B) \\
&- D_{M_2,P_1} \cos(A)\sin(A)\cos(B)\sin(B) \\
&- D_{M_1,P_2} \cos(A)\sin(A)\cos(B)\sin(B) + D_{M_3,P_2} \sin^2(A)\sin(B) \\
&- D_{M_2,P_3} \cos^2(A)\sin(B) + D_{M_1,P_1} \cos(A)\sin(A)\cos^2(B) \\
&- D_{M_3,P_1} \sin^2(A)\cos(B) + D_{M_1,P_3} \cos^2(A)\cos(B) \\
&- D_{M_3,P_3} \cos(A)\sin(A))DEL(P_3) - (-D_{M_2,U_1} \cos(A)\sin^2(B) \\
&- D_{M_2,U_2} \cos(A)\cos(B)\sin(B) + D_{M_1,U_1} \cos(A)\cos(B)\sin(B) \\
&- D_{M_3,U_1} \sin(A)\sin(B) + D_{M_1,U_2} \cos(A)\cos^2(B) \\
&- D_{M_3,U_2} \sin(A)\cos(B))DEL(U_2) - (-D_{M_2,P_1} \cos(A)\sin^2(B) \\
&- D_{M_2,P_2} \cos(A)\cos(B)\sin(B) + D_{M_1,P_1} \cos(A)\cos(B)\sin(B) \\
&- D_{M_3,P_1} \sin(A)\sin(B) + D_{M_1,P_2} \cos(A)\cos^2(B)
\end{aligned}
$$

$$- D_{M_3,P_2} \text{SIN(A)COS(B))DEL}(P_2) - (D_{M_2,U_2} \text{COS}^2\text{(A)SIN}^2\text{(B)}$$

$$- D_{M_2,U_1} \text{COS}^2\text{(A)COS(B)SIN(B)} - D_{M_1,U_2} \text{COS}^2\text{(A)COS(B)SIN(B)}$$

$$+ D_{M_3,U_2} \text{COS(A)SIN(A)SIN(B)} + D_{M_2,U_3} \text{COS(A)SIN(A)SIN(B)}$$

$$+ D_{M_1,U_1} \text{COS}^2\text{(A)COS}^2\text{(B)} - D_{M_3,U_1} \text{COS(A)SIN(A)COS(B)}$$

$$- D_{M_1,U_3} \text{COS(A)SIN(A)COS(B)} + D_{M_3,U_3} \text{SIN}^2\text{(A))DEL}(U_1)$$

$$- (D_{M_2,P_2} \text{COS}^2\text{(A)SIN}^2\text{(B)} - D_{M_2,P_1} \text{COS}^2\text{(A)COS(B)SIN(B)}$$

$$- D_{M_1,P_2} \text{COS}^2\text{(A)COS(B)SIN(B)} + D_{M_3,P_2} \text{COS(A)SIN(A)SIN(B)}$$

$$+ D_{M_2,P_3} \text{COS(A)SIN(A)SIN(B)} + D_{M_1,P_1} \text{COS}^2\text{(A)COS}^2\text{(B)}$$

$$- D_{M_3,P_1} \text{COS(A)SIN(A)COS(B)} - D_{M_1,P_3} \text{COS(A)SIN(A)COS(B)}$$

$$+ D_{M_3,P_3} \text{SIN}^2\text{(A))DEL}(P_1) + S_{M_2} \text{COS(A)SIN(B)} - S_{M_1} \text{COS(A)COS(B)}$$

$$+ S_{M_3} \text{SIN(A)} + J_{1,3}(\frac{d}{dT} P_3) + J_{1,2}(\frac{d}{dT} P_2) + J_{1,1}(\frac{d}{dT} P_1)$$

$$+ P_2(P_3 J_{3,3} + P_2 J_{3,2} + P_1 J_{3,1}) - P_3(J_{2,3} P_3 + P_2 J_{2,2} + P_1 J_{2,1})$$

$$TM_2 = -(D_{M_1,C_K} \text{SIN(B)} + D_{M_2,C_K} \text{COS(B))DEL}(C_K)$$

$$- (-D_{M_1,U_2} \text{SIN(A)SIN}^2\text{(B)} - D_{M_2,U_2} \text{SIN(A)COS(B)SIN(B)}$$

$$+ D_{M_1,U_1} \text{SIN(A)COS(B)SIN(B)} + D_{M_1,U_3} \text{COS(A)SIN(B)}$$

$$+ D_{M_2,U_1} \text{SIN(A)COS}^2\text{(B)} + D_{M_2,U_3} \text{COS(A)COS(B))DEL}(U_3)$$

$$- (-D_{M_1,P_2} \text{SIN(A)SIN}^2\text{(B)} - D_{M_2,P_2} \text{SIN(A)COS(B)SIN(B)}$$

$$+ D_{M_1,P_1} \text{ SIN(A)COS(B)SIN(B)} + D_{M_1,P_3} \text{ COS(A)SIN(B)}$$

$$+ D_{M_2,P_1} \text{ SIN(A)COS}^2\text{(B)} + D_{M_2,P_3} \text{ COS(A)COS(B))DEL}(P_3)$$

$$- (D_{M_1,U_1} \text{ SIN}^2\text{(B)} + D_{M_2,U_1} \text{ COS(B)SIN(B)} + D_{M_1,U_2} \text{ COS(B)SIN(B)}$$

$$+ D_{M_2,U_2} \text{ COS}^2\text{(B))DEL}(U_2) - (D_{M_1,P_1} \text{ SIN}^2\text{(B)} + D_{M_2,P_1} \text{ COS(B)SIN(B)}$$

$$+ D_{M_1,P_2} \text{ COS(B)SIN(B)} + D_{M_2,P_2} \text{ COS}^2\text{(B))DEL}(P_2)$$

$$- (-D_{M_1,U_2} \text{ COS(A)SIN}^2\text{(B)} - D_{M_2,U_2} \text{ COS(A)COS(B)SIN(B)}$$

$$+ D_{M_1,U_1} \text{ COS(A)COS(B)SIN(B)} - D_{M_1,U_3} \text{ SIN(A)SIN(B)}$$

$$+ D_{M_2,U_1} \text{ COS(A)COS}^2\text{(B)} - D_{M_2,U_3} \text{ SIN(A)COS(B))DEL}(U_1)$$

$$- (-D_{M_1,P_2} \text{ COS(A)SIN}^2\text{(B)} - D_{M_2,P_2} \text{ COS(A)COS(B)SIN(B)}$$

$$+ D_{M_1,P_1} \text{ COS(A)COS(B)SIN(B)} - D_{M_1,P_3} \text{ SIN(A)SIN(B)}$$

$$+ D_{M_2,P_1} \text{ COS(A)COS}^2\text{(B)} - D_{M_2,P_3} \text{ SIN(A)COS(B))DEL}(P_1) - S_{M_1} \text{ SIN(B)}$$

$$- S_{M_2} \text{ COS(B)} + J_{2,3}(\frac{d}{dT} P_3) + J_{2,2}(\frac{d}{dT} P_2) + J_{2,1}(\frac{d}{dT} P_1)$$

$$- P_1(P_3J_{3,3} + P_2J_{3,2} + P_1J_{3,1}) + P_3(J_{1,3}P_3 + J_{1,2}P_2 + P_1J_{1,1})$$

$$TM_3 = -(-D_{M_2,C_K} \text{ SIN(A)SIN(B)} + D_{M_1,C_K} \text{ SIN(A)COS(B)}$$

$$+ D_{M_3,C_K} \text{ COS(A))DEL}(C_K) - (D_{M_2,U_2} \text{ SIN}^2\text{(A)SIN}^2\text{(B)}$$

$$- D_{M_2,U_1} \text{ SIN}^2\text{(A)COS(B)SIN(B)} - D_{M_1,U_2} \text{ SIN}^2\text{(A)COS(B)SIN(B)}$$

$$- D_{M_3,U_2} \text{ COS(A)SIN(A)SIN(B)} - D_{M_2,U_3} \text{ COS(A)SIN(A)SIN(B)}$$

$$+ D_{M_1,U_1} \text{SIN}^2(A)\text{COS}^2(B) + D_{M_3,U_1} \text{COS}(A)\text{SIN}(A)\text{COS}(B)$$

$$+ D_{M_1,U_3} \text{COS}(A)\text{SIN}(A)\text{COS}(B) + D_{M_3,U_3} \text{COS}^2(A))\text{DEL}(U_3)$$

$$- (D_{M_2,P_2} \text{SIN}^2(A)\text{SIN}^2(B) - D_{M_2,P_1} \text{SIN}^2(A)\text{COS}(B)\text{SIN}(B)$$

$$- D_{M_1,P_2} \text{SIN}^2(A)\text{COS}(B)\text{SIN}(B) - D_{M_3,P_2} \text{COS}(A)\text{SIN}(A)\text{SIN}(B)$$

$$- D_{M_2,P_3} \text{COS}(A)\text{SIN}(A)\text{SIN}(B) + D_{M_1,P_1} \text{SIN}^2(A)\text{COS}^2(B)$$

$$+ D_{M_3,P_1} \text{COS}(A)\text{SIN}(A)\text{COS}(B) + D_{M_1,P_3} \text{COS}(A)\text{SIN}(A)\text{COS}(B)$$

$$+ D_{M_3,P_3} \text{COS}^2(A))\text{DEL}(P_3) - (-D_{M_2,U_1} \text{SIN}(A)\text{SIN}^2(B)$$

$$- D_{M_2,U_2} \text{SIN}(A)\text{COS}(B)\text{SIN}(B) + D_{M_1,U_1} \text{SIN}(A)\text{COS}(B)\text{SIN}(B)$$

$$+ D_{M_3,U_1} \text{COS}(A)\text{SIN}(B) + D_{M_1,U_2} \text{SIN}(A)\text{COS}^2(B)$$

$$+ D_{M_3,U_2} \text{COS}(A)\text{COS}(B))\text{DEL}(U_2) - (-D_{M_2,P_1} \text{SIN}(A)\text{SIN}^2(B)$$

$$- D_{M_2,P_2} \text{SIN}(A)\text{COS}(B)\text{SIN}(B) + D_{M_1,P_1} \text{SIN}(A)\text{COS}(B)\text{SIN}(B)$$

$$+ D_{M_3,P_1} \text{COS}(A)\text{SIN}(B) + D_{M_1,P_2} \text{SIN}(A)\text{COS}^2(B)$$

$$+ D_{M_3,P_2} \text{COS}(A)\text{COS}(B))\text{DEL}(P_2) - (D_{M_2,U_2} \text{COS}(A)\text{SIN}(A)\text{SIN}^2(B)$$

$$- D_{M_2,U_1} \text{COS}(A)\text{SIN}(A)\text{COS}(B)\text{SIN}(B)$$

$$- D_{M_1,U_2} \text{COS}(A)\text{SIN}(A)\text{COS}(B)\text{SIN}(B) + D_{M_2,U_3} \text{SIN}^2(A)\text{SIN}(B)$$

$$- D_{M_3,U_2} \text{COS}^2(A)\text{SIN}(B) + D_{M_1,U_1} \text{COS}(A)\text{SIN}(A)\text{COS}^2(B)$$

$$- D_{M_1,U_3} \text{SIN}^2(A)\text{COS}(B) + D_{M_3,U_1} \text{COS}^2(A)\text{COS}(B)$$

$$- D_{M_3,U_3} \cos(A)\sin(A))\mathrm{DEL}(U_1) - (D_{M_2,P_2} \cos(A)\sin(A)\sin^2(B)$$

$$- D_{M_2,P_1} \cos(A)\sin(A)\cos(B)\sin(B)$$

$$- D_{M_1,P_2} \cos(A)\sin(A)\cos(B)\sin(B) + D_{M_2,P_3} \sin^2(A)\sin(B)$$

$$- D_{M_3,P_2} \cos^2(A)\sin(B) + D_{M_1,P_1} \cos(A)\sin(A)\cos^2(B)$$

$$- D_{M_1,P_3} \sin^2(A)\cos(B) + D_{M_3,P_1} \cos^2(A)\cos(B)$$

$$- D_{M_3,P_3} \cos(A)\sin(A))\mathrm{DEL}(P_1) + S_{M_2} \sin(A)\sin(B)$$

$$- S_{M_1} \sin(A)\cos(B) - S_{M_3} \cos(A) + J_{3,3}(\frac{d}{dT} P_3) + J_{3,2}(\frac{d}{dT} P_2)$$

$$+ J_{3,1}(\frac{d}{dT} P_1) + P_1(J_{2,3}P_3 + P_2J_{2,2} + P_1J_{2,1})$$

$$- P_2(J_{1,3}P_3 + J_{1,2}P_2 + P_1J_{1,1})$$

Special Forms of the Moment Equation

As in the case of the force equations, the investigator sometimes wishes to modify the moment equations to determine how the vehicle will respond if the motion is restricted in some way. For the case of zero sideslip, MACSYMA goes through the equations, makes the appropriate substitutions, and displays the modified results. The zero sideslip condition requires that SIN(B) = 0 and COS(B) = 1. The substitution and display statements required to implement this assumption and the modified moment equations assume the following form:

```
(C33) FOR I:1 THRU 3 DO
 TM[I]:SUBST([SIN(B)=0,COS(B)=1],TM[I])$

(C34) FOR I:1 THRU 3 DO DISPLAY(TM[I])$
```

$$TM_1 = -(D_{M_1,C_K} \cos(A) - D_{M_3,C_K} \sin(A))\mathrm{DEL}(C_K)$$

$$- (-D_{M_3,U_1} \sin^2(A) - D_{M_3,U_3} \cos(A)\sin(A) + D_{M_1,U_1} \cos(A)\sin(A)$$

$$+ D_{M_1,U_3} \cos^2(A))\mathrm{DEL}(U_3) - (-D_{M_3,P_1} \sin^2(A) - D_{M_3,P_3} \cos(A)\sin(A)$$

$$+ D_{M_1,P_1} \text{COS(A)SIN(A)} + D_{M_1,P_3} \text{COS}^2\text{(A))DEL}(P_3) - (D_{M_1,U_2} \text{COS(A)}$$

$$- D_{M_3,U_2} \text{SIN(A))DEL}(U_2) - (D_{M_1,P_2} \text{COS(A)} - D_{M_3,P_2} \text{SIN(A))DEL}(P_2)$$

$$- (D_{M_3,U_3} \text{SIN}^2\text{(A)} - D_{M_3,U_1} \text{COS(A)SIN(A)} - D_{M_1,U_3} \text{COS(A)SIN(A)}$$

$$+ D_{M_1,U_1} \text{COS}^2\text{(A))DEL}(U_1) - (D_{M_3,P_3} \text{SIN}^2\text{(A)} - D_{M_3,P_1} \text{COS(A)SIN(A)}$$

$$- D_{M_1,P_3} \text{COS(A)SIN(A)} + D_{M_1,P_1} \text{COS}^2\text{(A))DEL}(P_1) + S_{M_3} \text{SIN(A)}$$

$$- S_{M_1} \text{COS(A)} + J_{1,3}(\frac{d}{dT} P_3) + J_{1,2}(\frac{d}{dT} P_2) + J_{1,1}(\frac{d}{dT} P_1)$$

$$+ P_2(P_3J_{3,3} + P_2J_{3,2} + P_1J_{3,1}) - P_3(J_{2,3}P_3 + P_2J_{2,2} + P_1J_{2,1})$$

$$TM_2 = -D_{M_2,C_K} \text{DEL}(C_K) - (D_{M_2,U_1} \text{SIN(A)} + D_{M_2,U_3} \text{COS(A))DEL}(U_3)$$

$$- (D_{M_2,P_1} \text{SIN(A)} + D_{M_2,P_3} \text{COS(A))DEL}(P_3) - D_{M_2,U_2} \text{DEL}(U_2)$$

$$- D_{M_2,P_2} \text{DEL}(P_2) - (D_{M_2,U_1} \text{COS(A)} - D_{M_2,U_3} \text{SIN(A))DEL}(U_1)$$

$$- (D_{M_2,P_1} \text{COS(A)} - D_{M_2,P_3} \text{SIN(A))DEL}(P_1) + J_{2,3}(\frac{d}{dT} P_3)$$

$$+ J_{2,2}(\frac{d}{dT} P_2) - S_{M_2} + J_{2,1}(\frac{d}{dT} P_1) - P_1(P_3J_{3,3} + P_2J_{3,2} + P_1J_{3,1})$$

$$+ P_3(J_{1,3}P_3 + J_{1,2}P_2 + P_1J_{1,1})$$

$$TM_3 = -(D_{M_1,C_K} \text{SIN(A)} + D_{M_3,C_K} \text{COS(A))DEL}(C_K) - (D_{M_1,U_1} \text{SIN}^2\text{(A)}$$

$$+ D_{M_3,U_1} \text{COS(A)SIN(A)} + D_{M_1,U_3} \text{COS(A)SIN(A)}$$

$$+ D_{M_3,U_3} \text{COS}^2\text{(A))DEL}(U_3) - (D_{M_1,P_1} \text{SIN}^2\text{(A)} + D_{M_3,P_1} \text{COS(A)SIN(A)}$$

$$+ D_{M_1,P_3} \text{COS(A)SIN(A)} + D_{M_3,P_3} \text{COS}^2\text{(A))DEL}(P_3) - (D_{M_1,U_2} \text{SIN(A)}$$

$$+ D_{M_3,U_2} \cos(A))DEL(U_2) - (D_{M_1,P_2} \sin(A) + D_{M_3,P_2} \cos(A))DEL(P_2)$$

$$- (-D_{M_1,U_3} \sin^2(A) - D_{M_3,U_3} \cos(A)\sin(A) + D_{M_1,U_1} \cos(A)\sin(A)$$

$$+ D_{M_3,U_1} \cos^2(A))DEL(U_1) - (-D_{M_1,P_3} \sin^2(A) - D_{M_3,P_3} \cos(A)\sin(A)$$

$$+ D_{M_1,P_1} \cos(A)\sin(A) + D_{M_3,P_1} \cos^2(A))DEL(P_1) - S_{M_1} \sin(A)$$

$$- S_{M_3} \cos(A) + J_{3,3}(\frac{d}{dT} P_3) + J_{3,2}(\frac{d}{dT} P_2) + J_{3,1}(\frac{d}{dT} P_1)$$

$$+ P_1(J_{2,3}P_3 + P_2J_{2,2} + P_1J_{2,1}) - P_2(J_{1,3}P_3 + J_{1,2}P_2 + P_1J_{1,1})$$

In addition to the zero sideslip condition, the investigator frequently wishes to determine vehicle response when the angle of attack is limited to small values. For this condition MACSYMA would implement the assumption that SIN(A) = A. Moreover, if the angle of attack were sufficiently small, the program would request MACSYMA to make the additional substitution COS(A) = 1.

In this case, the required substitution and display statements give rise to the following modified moment equations:

```
(C35) FOR I:1 THRU 3 DO TM[I]:SUBST([SIN(A)=A,COS(A)=1],TM[I])$

(C36) FOR I:1 THRU 3 DO DISPLAY(TM[I])$
```

$$TM_1 = -(D_{M_1,C_K} - D_{M_3,C_K} A)DEL(C_K) - (-D_{M_3,U_1} A^2 - D_{M_3,U_3} A$$

$$+ D_{M_1,U_1} A + D_{M_1,U_3})DEL(U_3) - (-D_{M_3,P_1} A^2 - D_{M_3,P_3} A + D_{M_1,P_1} A$$

$$+ D_{M_1,P_3})DEL(P_3) - (D_{M_1,U_2} - D_{M_3,U_2} A)DEL(U_2) - (D_{M_1,P_2}$$

$$- D_{M_3,P_2} A)DEL(P_2) - (D_{M_3,U_3} A^2 - D_{M_3,U_1} A - D_{M_1,U_3} A$$

$$+ D_{M_1,U_1})DEL(U_1) - (D_{M_3,P_3} A - D_{M_3,P_1} A - D_{M_1,P_3} A$$

$$+ D_{M_1,P_1})DEL(P_1) + S_{M_3} A + J_{1,3}(\frac{d}{dT} P_3) + J_{1,2}(\frac{d}{dT} P_2) + J_{1,1}(\frac{d}{dT} P_1)$$

$$- S_{M_1} + P_2(P_3 J_{3,3} + P_2 J_{3,2} + P_1 J_{3,1}) - P_3(J_{2,3} P_3 + P_2 J_{2,2} + P_1 J_{2,1})$$

$$TM_2 = -D_{M_2,C_K}\ DEL(C_K) - (D_{M_2,U_1}\ A + D_{M_2,U_3})DEL(U_3) - (D_{M_2,P_1}\ A + D_{M_2,P_3})DEL(P_3) - D_{M_2,U_2}\ DEL(U_2) - D_{M_2,P_2}\ DEL(P_2) - (D_{M_2,U_1} - D_{M_2,U_3}\ A)DEL(U_1) - (D_{M_2,P_1} - D_{M_2,P_3}\ A)DEL(P_1) + J_{2,3}(\frac{d}{dT}\ P_3) + J_{2,2}(\frac{d}{dT}\ P_2) - S_{M_2} + J_{2,1}(\frac{d}{dT}\ P_1) - P_1(P_3 J_{3,3} + P_2 J_{3,2} + P_1 J_{3,1}) + P_3(J_{1,3} P_3 + J_{1,2} P_2 + P_1 J_{1,1})$$

$$TM_3 = -(D_{M_1,C_K}\ A + D_{M_3,C_K})DEL(C_K) - (D_{M_1,U_1}\ A^2 + D_{M_3,U_1}\ A + D_{M_1,U_3}\ A + D_{M_3,U_3})DEL(U_3) - (D_{M_1,P_1}\ A^2 + D_{M_3,P_1}\ A + D_{M_1,P_3}\ A + D_{M_3,P_3})DEL(P_3) - (D_{M_1,U_2}\ A + D_{M_3,U_2})DEL(U_2) - (D_{M_1,P_2}\ A + D_{M_3,P_2})DEL(P_2) - (-D_{M_1,U_3}\ A^2 - D_{M_3,U_3}\ A + D_{M_1,U_1}\ A + D_{M_3,U_1})DEL(U_1) - (-D_{M_1,P_3}\ A^2 - D_{M_3,P_3}\ A + D_{M_1,P_1}\ A + D_{M_3,P_1})DEL(P_1) - S_{M_1}\ A + J_{3,3}(\frac{d}{dT}\ P_3) - S_{M_3} + J_{3,2}(\frac{d}{dT}\ P_2) + J_{3,1}(\frac{d}{dT}\ P_1) + P_1(J_{2,3} P_3 + P_2 J_{2,2} + P_1 J_{2,1}) - P_2(J_{1,3} P_3 + J_{1,2} P_2 + P_1 J_{1,1})$$

Examination of these equations reveals the existence of terms such as $A^2$. If it is assumed that second-order terms in $A$ are negligible, a program statement instructing MACSYMA to make the substitution $A^2 = 0$ would simplify the moment equations as follows:

```
(C37) FOR I:1 THRU 3 DO
 TM[I]:SUBST([A**2=0],TM[I])$

(C39) FOR I:1 THRU 3 DO DISPLAY(TM[I])$
```

$$TM_1 = -(D_{M_1,C_K} - D_{M_3,C_K}\ A)DEL(C_K) - (-D_{M_3,U_3}\ A + D_{M_1,U_1}\ A$$
$$+ D_{M_1,U_3})DEL(U_3) - (-D_{M_3,P_3}\ A + D_{M_1,P_1}\ A + D_{M_1,P_3})DEL(P_3)$$
$$- (D_{M_1,U_2} - D_{M_3,U_2}\ A)DEL(U_2) - (D_{M_1,P_2} - D_{M_3,P_2}\ A)DEL(P_2)$$
$$- (-D_{M_3,U_1}\ A - D_{M_1,U_3}\ A + D_{M_1,U_1})DEL(U_1) - (-D_{M_3,P_1}\ A - D_{M_1,P_3}\ A$$
$$+ D_{M_1,P_1})DEL(P_1) + S_{M_3}\ A + J_{1,3}(\frac{d}{dT}\ P_3) + J_{1,2}(\frac{d}{dT}\ P_2)$$
$$+ J_{1,1}(\frac{d}{dT}\ P_1) - S_{M_1} + P_2(P_3J_{3,3} + P_2J_{3,2} + P_1J_{3,1})$$
$$- P_3(J_{2,3}P_3 + P_2J_{2,2} + P_1J_{2,1})$$

$$TM_2 = -D_{M_2,C_K}\ DEL(C_K) - (D_{M_2,U_1}\ A + D_{M_2,U_3})DEL(U_3) - (D_{M_2,P_1}\ A$$
$$+ D_{M_2,P_3})DEL(P_3) - D_{M_2,U_2}\ DEL(U_2) - D_{M_2,P_2}\ DEL(P_2) - (D_{M_2,U_1}$$
$$- D_{M_2,U_3}\ A)DEL(U_1) - (D_{M_2,P_1} - D_{M_2,P_3}\ A)DEL(P_1) + J_{2,3}(\frac{d}{dT}\ P_3)$$
$$+ J_{2,2}(\frac{d}{dT}\ P_2) - S_{M_2} + J_{2,1}(\frac{d}{dT}\ P_1) - P_1(P_3J_{3,3} + P_2J_{3,2} + P_1J_{3,1})$$
$$+ P_3(J_{1,3}P_3 + J_{1,2}P_2 + P_1J_{1,1})$$

$$TM_3 = -(D_{M_1,C_K}\ A + D_{M_3,C_K})DEL(C_K) - (D_{M_3,U_1}\ A + D_{M_1,U_3}\ A$$
$$+ D_{M_3,U_3})DEL(U_3) - (D_{M_3,P_1}\ A + D_{M_1,P_3}\ A + D_{M_3,P_3})DEL(P_3)$$
$$- (D_{M_1,U_2}\ A + D_{M_3,U_2})DEL(U_2) - (D_{M_1,P_2}\ A + D_{M_3,P_2})DEL(P_2)$$
$$- (-D_{M_3,U_3}\ A + D_{M_1,U_1}\ A + D_{M_3,U_1})DEL(U_1) - (-D_{M_3,P_3}\ A + D_{M_1,P_1}\ A$$
$$+ D_{M_3,P_1})DEL(P_1) - S_{M_1}\ A + J_{3,3}(\frac{d}{dT}\ P_3) - S_{M_3} + J_{3,2}(\frac{d}{dT}\ P_2)$$

$$+ J_{3,1}\left(\frac{d}{dT} P_1\right) + P_1(J_{2,3}P_3 + P_2J_{2,2} + P_1J_{2,1})$$

$$- P_2(J_{1,3}P_3 + J_{1,2}P_2 + P_1J_{1,1})$$

An additional simplification is possible if the assumption that angular velocity perturbations are negligible is a valid one. Implementation of the assumption that $DEL(P_i) = 0$ yields the following greatly simplified equations:

```
(C40) FOR I:1 THRU 3 DO FOR J:1 THRU 3 DO
 TM[I]:SUBST([DEL(P[J])=0],TM[I])$

(C41) FOR I:1 THRU 3 DO DISPLAY(TM[I])$
```

$$TM_1 = -(D_{M_1,C_K} - D_{M_3,C_K} A)DEL(C_K) - (-D_{M_3,U_3} A + D_{M_1,U_1} A$$

$$+ D_{M_1,U_3})DEL(U_3) - (D_{M_1,U_2} - D_{M_3,U_2} A)DEL(U_2) - (-D_{M_3,U_1} A$$

$$- D_{M_1,U_3} A + D_{M_1,U_1})DEL(U_1) + S_{M_3} A + J_{1,3}\left(\frac{d}{dT} P_3\right) + J_{1,2}\left(\frac{d}{dT} P_2\right)$$

$$+ J_{1,1}\left(\frac{d}{dT} P_1\right) - S_{M_1} + P_2(P_3J_{3,3} + P_2J_{3,2} + P_1J_{3,1})$$

$$- P_3(J_{2,3}P_3 + P_2J_{2,2} + P_1J_{2,1})$$

$$TM_2 = -D_{M_2,C_K} DEL(C_K) - (D_{M_2,U_1} A + D_{M_2,U_3})DEL(U_3)$$

$$- D_{M_2,U_2} DEL(U_2) - (D_{M_2,U_1} - D_{M_2,U_3} A)DEL(U_1) + J_{2,3}\left(\frac{d}{dT} P_3\right)$$

$$+ J_{2,2}\left(\frac{d}{dT} P_2\right) - S_{M_2} + J_{2,1}\left(\frac{d}{dT} P_1\right) - P_1(P_3J_{3,3} + P_2J_{3,2} + P_1J_{3,1})$$

$$+ P_3(J_{1,3}P_3 + J_{1,2}P_2 + P_1J_{1,1})$$

$$TM_3 = -(D_{M_1,C_K} A + D_{M_3,C_K})DEL(C_K) - (D_{M_3,U_1} A + D_{M_1,U_3} A$$

$$+ D_{M_3,U_3})DEL(U_3) - (D_{M_1,U_2} A + D_{M_3,U_2})DEL(U_2) - (-D_{M_3,U_3} A$$

$$+ D_{M_1,U_1} A + D_{M_3,U_1})DEL(U_1) - S_{M_1} A + J_{3,3}\left(\frac{d}{dT} P_3\right) - S_{M_3}$$

$$+ J_{3,2}\left(\frac{d}{dT} P_2\right) + J_{3,1}\left(\frac{d}{dT} P_1\right) + P_1(J_{2,3}P_3 + P_2J_{2,2} + P_1J_{2,1})$$

$$- P_2(J_{1,3}P_3 + J_{1,2}P_2 + P_1J_{1,1})$$

Thrust Moments

As indicated previously, the thrust moments $TM_i$ appearing on the left-hand side of these equations are the resultant of the moments produced by a number of thrust generating systems. Equations (13) relate the thrust axes coordinates $X_n{}^i$ to the coordinate system $Y_n{}^i$, which has the same origin as the thrust axes but is parallel to the body axes systems.

To facilitate the formulation, equations (13) are entered here.

```
(C1) Y[1,N]:X[1,N]*COS(K[N])*COS(P[N])$

(C2) Y[2,N]:X[1,N]*COS(K[N])*SIN(P[N])$

(C3) Y[3,N]:-X[1,N]*SIN(K[N])$
```

The point of application of the $n$th thrust vector relative to the body axes system, with origin at the center of gravity, has components $(L_{1,n}, L_{2,n}, L_{3,n})$. The components of the $n$th thrust vector in this coordinate system are given by equations (14). The product of the position matrix with elements $(L_{1,n}, L_{2,n}, L_{3,n})$ and a column vector of thrust components can be processed as follows.

First enter the (3.3) position matrix, element by element, as requested by MACSYMA. Next enter the (3.1) column vector of thrust components in the same manner. When the matrices are entered, the displayed form of each matrix assumes the conventional textbook form

```
(C4) ENTERMATRIX(3,3);

ROW 1 COLUMN 1  0;

ROW 1 COLUMN 2  -L[3,N];

ROW 1 COLUMN 3  L[2,N];

ROW 2 COLUMN 1  L[3,N];

ROW 2 COLUMN 2  0;

ROW 2 COLUMN 3  -L[1,N];

ROW 3 COLUMN 1  -L[2,N];

ROW 3 COLUMN 2  L[1,N];

ROW 3 COLUMN 3  0;
```

```
MATRIX-ENTERED
                          [    0      -L        L     ]
                          [              3,N     2,N  ]
(D4)                      [  L          0      -L     ]
                          [   3,N                1,N  ]
                          [ -L          L        0    ]
                          [   2,N        1,N          ]

(C5) ENTERMATRIX(3,1);

ROW 1 COLUMN 1  T[N]*DIFF(Y[1,N],X[1,N]);

ROW 2 COLUMN 1  T[N]*DIFF(Y[2,N],X[1,N]);

ROW 3 COLUMN 1  T[N]*DIFF(Y[3,N],X[1,N]);

MATRIX-ENTERED
                          [ T  COS(K ) COS(P ) ]
                          [  N      N       N  ]
(D5)                      [ T  COS(K ) SIN(P ) ]
                          [  N      N       N  ]
                          [     -T  SIN(K )    ]
                          [       N      N     ]
```

By requesting the system to multiply these two matrices, the following product matrix is obtained:

```
(C6) (D4).(D5);

       [       -L    T  COS(K ) SIN(P ) - L    T  SIN(K )      ]
       [         3,N  N      N       N     2,N  N      N       ]
(D6)   [        L    T  COS(K ) COS(P ) + L    T  SIN(K )      ]
       [         3,N  N      N       N     1,N  N      N       ]
       [ L    T  COS(K ) SIN(P ) - L    T  COS(K ) COS(P )     ]
       [  1,N  N      N       N     2,N  N      N       N      ]
```

In order to express this column vector of thrust moments in conventional functional form, the following two programming steps are required:

```
(C7) FOR I:1 THRU 3 DO ROW[I]:FIRST(ROW((D6),I));

(C8) FOR I:1 THRU 3 DO (TM[I]:ROW[I][1],DISPLAY(TM[I]))$
```

$$TM_1 = -L_{3,N}\ T_N\ COS(K_N)\ SIN(P_N) - L_{2,N}\ T_N\ SIN(K_N)$$

$$TM_2 = L_{3,N}\ T_N\ COS(K_N)\ COS(P_N) + L_{1,N}\ T_N\ SIN(K_N)$$

$$TM_3 = L_{1,N}\ T_N\ COS(K_N)\ SIN(P_N) - L_{2,N}\ T_N\ COS(K_N)\ COS(P_N)$$

These equations give the moments produced by the $n$th thrust vector. When the number of thrust generating systems is known, these equations can be summed on $n$ to obtain the total thrust moments.

## Spatial Orientation in Terms of the Direction Cosines

The differential equations for the direction cosines can be obtained by first entering a (3,1) column vector of direction cosines, with elements $D_{I1}$, $D_{I2}$, and $D_{I3}$, where $I$ can assume the values 1,2,3, and by premultiplying this vector by the angular velocity matrix. This operation is equivalent to the vector cross product of the angular velocity vector and the unit vectors $\hat{I}$, $\hat{J}$, and $\hat{K}$. The programming steps and the displayed output are

```
(C1) ENTERMATRIX(3,1);

ROW 1 COLUMN 1  D[I,1];

ROW 2 COLUMN 1  D[I,2];

ROW 3 COLUMN 1  D[I,3];

MATRIX-ENTERED
                                    [ D    ]
                                    [  I,1 ]
                                    [      ]
(D1)                                [ D    ]
                                    [  I,2 ]
                                    [      ]
                                    [ D    ]
                                    [  I,3 ]

(C2) ENTERMATRIX(3,3);

ROW 1 COLUMN 1  0;

ROW 1 COLUMN 2  -P[3];

ROW 1 COLUMN 3  P[2];

ROW 2 COLUMN 1  P[3];

ROW 2 COLUMN 2  0;

ROW 2 COLUMN 3  -P[1];

ROW 3 COLUMN 1  -P[2];

ROW 3 COLUMN 2  P[1];

ROW 3 COLUMN 3  0;
```

MATRIX-ENTERED

(D2)
$$\begin{bmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}$$

The product of these two matrices is

```
(C3) (D2).(D1);
```

(D3)
$$\begin{bmatrix} P_2 D_{I,3} - P_3 D_{I,2} \\ P_3 D_{I,1} - P_1 D_{I,3} \\ P_1 D_{I,2} - P_2 D_{I,1} \end{bmatrix}$$

The individual terms of this column vector can be evaluated for $I = 1,2,3$ by executing the following program statement:

```
(C4) FOR I:1 THRU 3 DO FOR J:1 THRU 3 DO
 EV(C[I,J]:ROW((D3),J))$
```

The evaluated terms can be printed out by using the now familiar display statement

```
(C5) FOR I:1 THRU 3 DO FOR J:1 THRU 3 DO DISPLAY(C[I,J])$
```

$$C_{1,1} = [D_{1,3}\, P_2 - D_{1,2}\, P_3]$$

$$C_{1,2} = [D_{1,1}\, P_3 - P_1\, D_{1,3}]$$

$$C_{1,3} = [P_1\, D_{1,2} - D_{1,1}\, P_2]$$

$$C_{2,1} = [P_2\, D_{2,3} - D_{2,2}\, P_3]$$

$$C_{2,2} = [D_{2,1}\, P_3 - P_1\, D_{2,3}]$$

$$C_{2,3} = [P_1\, D_{2,2} - P_2\, D_{2,1}]$$

$$C_{3,1} = [P_2\, D_{3,3} - P_3\, D_{3,2}]$$

$$C_{3,2} = [P_3\, D_{3,1} - P_1\, D_{3,3}]$$

$$C_{3,3} = [P_1\, D_{3,2} - P_2\, D_{3,1}]$$

The dependence of the direction cosines on the indices $I$ and $J$ and the time $T$ can be shown by using the DEPENDENCIES statement. The use of this statement facilitates the formulation of the differential coefficients

```
(C6) DEPENDENCIES(D(I,J,T))$
```

It only remains to request that the differential coefficients of the direction cosines $DC_{IJ}$ with respect to the time $T$ be added to the coefficients $C_{IJ}$ and displayed as follows:

```
(C7) FOR I:1 THRU 3 DO FOR J:1 THRU 3 DO
 DC[I,J]:C[I,J]+DIFF(D[I,J],T)$

(C8) FOR I:1 THRU 3 DO FOR J:1 THRU 3 DO DISPLAY(DC[I,J])$
```

$$DC_{1,1} = [D_{1,3}\,P_2 - D_{1,2}\,P_3] + \frac{d}{dT} D_{1,1} = 0$$

$$DC_{1,2} = [D_{1,1}\,P_3 - P_1\,D_{1,3}] + \frac{d}{dT} D_{1,2} = 0$$

$$DC_{1,3} = [P_1\,D_{1,2} - D_{1,1}\,P_2] + \frac{d}{dT} D_{1,3} = 0$$

$$DC_{2,1} = [P_2\,D_{2,3} - D_{2,2}\,P_3] + \frac{d}{dT} D_{2,1} = 0$$

$$DC_{2,2} = [D_{2,1}\,P_3 - P_1\,D_{2,3}] + \frac{d}{dT} D_{2,2} = 0$$

$$DC_{2,3} = [P_1\,D_{2,2} - P_2\,D_{2,1}] + \frac{d}{dT} D_{2,3} = 0$$

$$DC_{3,1} = [P_2\,D_{3,3} - P_3\,D_{3,2}] + \frac{d}{dT} D_{3,1} = 0$$

$$DC_{3,2} = [P_3\,D_{3,1} - P_1\,D_{3,3}] + \frac{d}{dT} D_{3,2} = 0$$

$$DC_{3,3} = [P_1\,D_{3,2} - P_2\,D_{3,1}] + \frac{d}{dT} D_{3,3} = 0$$

This concludes the formulation of the simplified aeronautical model considered. The formulation gave rise to 18 equations: 3 force equations; 3 moment equations; 9 direction cosine equations to determine the spatial orientation of the vehicle; and 3 equations to determine the geographical location of the vehicle relative to an Earth-fixed reference frame. It is seen that the technique of symbolic mathematical computation, as implemented by the MACSYMA system, can be used to facilitate the formulation of complex mathematical models of physical systems and reduce the errors to which human operators are prone. The versatility and simplicity of the system make it attractive to programmers and nonprogrammers alike. Moreover, as already noted, the capability of working interactively enhances the utility of the system by permitting the user to modify the formulation as he proceeds.

Ames Research Center
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Moffett Field, California 94035, January 12, 1979

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16. Abstract

A mathematical model of a dynamical system describes the physical characteristics of the system and can be used to determine the response of the system to the forces encountered. In the case of aeronautical systems exposed to gravity, inertia, aerodynamic, and thrust forces, the mathematical model enables the engineer to determine the state of the vehicle, its spatial orientation, and the geographical location as a function of time. Sometimes the vehicle response is determined by direct calculation; at other times, simulators are used. The simulator moves in response to the computed solutions of the mathematical model equations. These solutions represent the response of the system to the forcing functions generated by the simulator pilot's control inputs. The formulation of mathematical models of complex multidegrees-of-freedom dynamical systems is time consuming and subject to human error. In view of the complexity involved, it is desirable to mechanize as much of the formulation as possible. Recent developments in formula manipulation compilers have led to an extension of the area of application of digital computers beyond the purely numerical data processing stage. These developments, combined with the design of several symbol manipulation languages, enable computers to be used for symbolic mathematical computation. This technique provides for the symbolic manipulation of mathematical expressions: for example, the expression SIN(X) can be differentiated, resulting in the expression COS(X). Moreover, it can be used frequently to obtain symbolic solutions in problem areas that heretofore could only be approached numerically. A computer system and language that can be used to perform symbolic manipulations in an interactive mode have been used to formulate a mathematical model of an aeronautical system. The example demonstrates that once the procedure is established, the formulation and modification of models for simulation studies can be reduced to a series of routine computer operations.

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